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Geometry of the gauge algebra in non-commutative Yang-Mills theory

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Abstract: A detailed description of the infinite-dimensional Lie algebra of $\star$-gauge transformations in non-commutative Yang-Mills theory is presented. Various descriptions of this algebra are given in terms of inner automorphisms of the underlying deformed algebra of functions on spacetime, of deformed symplectic diffeomorphisms, of the infinite unitary Lie algebra $\mathrm{u}(\infty)$, and of the C^*-algebra of compact operators on a quantum mechanical Hilbert space. The spacetime and string interpretations are also elucidated.

Keywords: Space-Time Symmetries, Gauge Symmetry, Non-Commutative Geometry.

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Contents

1. Introduction

One of the most interesting aspects of Yang-Mills theory on a non-commutative space [1]–[4] is its extended gauge symmetry. This symmetry group mixes the internal gauge degrees of freedom with the geometrical degrees of freedom in spacetime. There are several interesting consequences of this feature. The most striking one is that non-commutative gauge theory does not contain any local observables in the usual sense, because ordinary traces must be accompanied by an integration over

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the space in order to render operators gauge invariant. On the other hand, the non-commutative gauge symmetry seemingly allows for a larger class of observables. In addition to the usual loop observables, there are gauge invariant Wilson line operators associated with open contours which may be thought of as carrying a non-vanishing momentum [5, 6, 7]. From these objects it is possible to construct gauge invariant operators which carry definite momentum, and which reduce to the usual local gauge invariant operators of ordinary gauge field theory in the commutative limit [8]. In the D-brane picture, these gauge invariant operators naturally couple to general background supergravity fields [9].

The construction of gauge invariant observables associated with open Wilson lines relies on the translational invariance that persists in non-commutative gauge theories [7, 8]. The key observation is that translations in the non-commutative directions are equivalent (up to global symmetry transformations) to gauge transformations, i.e. they can be realized via conjugation by unitary elements of the gauge symmetry group. The only other theory that possesses such a property is general relativity, although in that case it is not so straightforward to write down a set of non-local invariant observables. In this paper we shall explore this feature further and determine to what extent non-commutative Yang-Mills theory may be regarded as a model of general relativity, i.e. as a theory whose local gauge symmetry includes general coordinate transformations.

There are several hints that non-commutative gauge theories on *flat* space naturally possess general covariance.

- In string theory, the unitary group of a closed string vertex operator algebra contains generic reparametrizations of the target space coordinates [10]. From this fact it is possible to realize general covariance as a gauge symmetry by using the chiral structure of the closed strings.

- Non-commutative gauge theories arise most naturally in string theory. In particular, they are typically defined on closed string backgrounds of the form $\mathcal{M} \times \mathbb{R}^d$, where $\mathcal{M}$ is the commuting worldvolume of a D-brane. The effect of turning on a non-degenerate Neveu-Schwarz B-field along the transverse space $\mathbb{R}^d$ can be absorbed, in a particular low-energy limit, into a description of the dynamics in terms of non-commutative gauge theory [2, 3]. As such, these field theories possess many stringy structures, but within a much simpler setting. For example, the one-loop long-ranged potential particular to non-commutative Yang-Mills theory can be identified with the gravitational interaction in type-IIB superstring theory [11, 12].

- Certain large-N matrix models provide a concrete, non-perturbative definition of non-commutative Yang-Mills theory [6, 11], and these discrete models have a $\mathrm{U}(\infty)$ symmetry group which may be identified with the group of area-

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preserving diffeomorphisms in two-dimensions [13]. It is natural then to expect that non-commutative gauge symmetries contain at least the subgroup of symplectic diffeomorphisms of the spacetime.

- In the large-N dual supergravity description of non-commutative Yang-Mills theory in four dimensions with 16 supersymmetries, there is a massless bound state which gives rise to the Newtonian gravitational force law [14].
- Via certain dimensional reduction techniques, the translational symmetry of non-commutative Yang-Mills theory can be gauged to induce a field theory which contains as special limits some gauge models of gravitation [15].

Other indications that noncommutativity implies general covariance can be found in [16, 17]. It therefore seems likely that non-commutative Yang-Mills theory contains both gauge theory and gravitation, and so is a good candidate for a unified and potentially renormalizable theory of the fundamental interactions including gravity.

To investigate these features further, in this paper we will give a detailed, analytic description of the Lie algebra of local non-commutative gauge transformations. This is particularly important for the interpretation of non-commutative Yang-Mills theory as some sort of gauge model of spacetime symmetries, because it admits a dual interpretation as ordinary Yang-Mills theory with this extended, infinite-dimensional gauge symmetry algebra [18]. At the same time this analysis will generally sharpen the present understanding of the local structure of the non-commutative symmetry groups underlying these gauge theories. We shall see in fact that many of the non local, stringy features of non-commutative field theories are captured by the geometrical properties of their symmetry groups. We will find that this infinite-dimensional algebra is a deformation of the Lie algebra of symplectic transformations of the space in a very precise way, the symplectic structure being given by the noncommutativity parameters. A similar sort of geometric description of the non-commutative gauge algebra has been given recently in [19]. We will also discuss the connections between this algebra and the infinite unitary Lie algebra $\mathrm{u}(\infty)$ which is the natural symmetry algebra from the point of view of the twisted reduced models describing non-commutative gauge theory. As we shall see, this description agrees with the recent proposal in [20], where global aspects of the non-commutative gauge group are described. Some other properties of the non-commutative gauge algebra have been recently discussed in [21].

We will start in the next section by describing the relations between gauge transformations and the inner automorphisms of the deformed algebra of functions on spacetime. For this, we shall define the non-commutative gauge symmetry by its finite representation on fundamental matter fields (similar arguments appear in [22]) and rigorously derive from this the corresponding infinitesimal transformation rules. A corollary of this derivation is that the fundamental representation is only irreducible in the Schrödinger polarization of the non-commutative function algebra. This fact immediately implies the description of [20] in terms of the algebra of compact operators acting on a single-particle, quantum mechanical Hilbert space, and it also agrees with the recent observations of [21] concerning the irreducibility of the matrix representations of the non-commutative gauge algebra.

In section 3 we explicitly describe properties of the gauge Lie algebra, and calculate its structure constants. This somewhat technical analysis is carried out in two different bases, and the relations between the two representations are described. We point out various physical characteristics of the algebras that we derive. In section 4 we will first describe how the gauge Lie algebra is a deformation of the Poisson-Lie algebra of symplectic diffeomorphisms, to which it reduces at the leading non-trivial order in the commutative limit. We then describe the intimate relationships between this algebra and the infinite-dimensional Lie algebra $\mathrm{u}(\infty)$. This leads explicitly to a description of the gauge algebra in terms of compact operators, as in [20], and lends some insight into the relationships between these gauge models, reduced models, and membrane physics. The relationships between such geometric algebras and $\mathrm{u}(\infty)$ have also been analysed recently in [23].

In section 5 we then describe various subalgebras of the gauge algebra, both finite and infinite dimensional, and interpret them as geometric symmetry transformations of the spacetime by examining the corresponding gauge transformations of fields that they induce. In section 6 we briefly present some applications of the technical formalism given in most of the paper. We discuss some more algebraic aspects of the representation of global spacetime coordinate transformations as gauge symmetries, and exemplify the fact that not all diffeomorphisms are realizable as inner automorphisms, but rather only the symplectic ones. We also comment on how the description of the gauge algebra in terms of compact operators is particularly well-suited to describe some aspects of solitons in non-commutative field theory, although a more thorough investigation requires dealing with topological aspects of the non-commutative gauge symmetry group (as in [20]) which lies outside the scope of this paper whose aim is to concentrate on the geometrical properties of the *local* gauge group. Finally, in section 7 we close with some concluding remarks.

2. Inner automorphisms in non-commutative gauge theory

In this section we will introduce some relevant concepts that will be used throughout this paper. We start by introducing Yang-Mills theory based on a non-commutative function algebra, and then describe the relationship between the automorphisms of this algebra and non-commutative gauge transformations.

2.1 Gauge theory on non-commutative space

In the simplest setting, a non-commutative space is defined as a space whose coordinates x_i, $i = 1, \ldots, d$, generate the Lie algebra

$$[x_i, x_j] = i\,\theta_{ij}\,, \tag{2.1}$$

where θ_{ij} is a constant antisymmetric real-valued matrix of length dimension 2. The algebra $\mathcal{A}_\theta$ of functions on this space is the algebra generated by the x_i,

$$\mathcal{A}_\theta = \frac{\mathcal{S}\left(\mathbb{R}^d\right)}{\mathcal{R}_\theta}\,, \tag{2.2}$$

where $\mathcal{R}_\theta$ stands for the commutation relations (2.1), and $\mathcal{S}(\mathbb{R}^d)$ denotes an appropriate Schwartz space of functions on $\mathbb{R}^d \to \mathbb{C}$ of rapid decrease at infinity which we regard as a subspace in the closure of the ring $\mathbb{C}[[x_1, \ldots, x_d]]$ of formal power series in the x_i. Later on we will be interested in appropriate C^*-completions of (2.2). The algebra $\mathcal{A}_\theta$ can be described as a deformation of the algebra $\mathcal{A}_0$ of ordinary, continuous functions on $\mathbb{R}^d \to \mathbb{C}$. In the following we will also restrict the functions of $\mathcal{A}_0$ to those which lie in the appropriate (dense) Schwartz space $\mathcal{S}(\mathbb{R}^d)$. The Banach norm on $\mathcal{A}_0$ is the usual L^∞-norm,

$$\|f\|_\infty = \sup_{x\in\mathbb{R}^d}\,|f(x)|\,, \qquad f \in \mathcal{A}_0\,. \tag{2.3}$$

For simplicity, we will assume in this paper that the (euclidean) spacetime dimension d is even and that θ_{ij} is of maximal rank. Otherwise, the algebra $\mathcal{A}_\theta$ has a non-trivial centre which we can quotient out to effectively induce a non-degenerate deformation matrix. Geometrically, this operation corresponds to the identification of an ordinary, commutative subspace of non-commutative $\mathbb{R}^d$.

The deformation is described by the Groenewold-Moyal $\star$-product [24, 25]

$$(f\star g)(x) = \left.\mathrm{e}^{\frac{i}{2}\theta_{ij}\,\partial_{\xi_i}\partial_{\eta_j}}\,f(\xi)g(\eta)\right|_{\xi=\eta=x}\,, \qquad f,g \in \mathcal{A}_0\,, \tag{2.4}$$

so that when $\theta_{ij} = 0$ the product reduces to the ordinary pointwise multiplication of functions, while at higher orders in θ_{ij} derivatives of the functions appear. In this sense the $\star$-product is non local. By integrating (2.4) by parts over $\mathbb{R}^d$ we see that it possesses the cyclic integration property

$$\int \mathrm{d}^dx\,(f_1\star\cdots\star f_n)\,(x) = \int \mathrm{d}^dx\,\left(f_{\pi(1)}\star\cdots\star f_{\pi(n)}\right)(x) \tag{2.5}$$

for any collection of functions $f_1, \ldots, f_n \in \mathcal{A}_0$ and any cyclic permutation $\pi \in S_n$.

Just as in the commutative case, we can introduce a linear derivation ∂_i of the algebra $\mathcal{A}_\theta$ by defining

$$\partial_i\,x_j = \delta_{ij}\,, \tag{2.6}$$

and then extending it to all of $\mathcal{A}_\theta$ using the usual Leibnitz rule. The derivation ∂_i can be represented as the $\star$-commutator with an element of $\mathbb{C}[[x_1, \ldots, x_d]]$,

$$\partial_i f = -i\,\theta_i^{\ j}\,[x_j, f]_\star\,, \tag{2.7}$$

where

$$\theta_i^{\ k}\,\theta_{kj} = \delta_{ij}\,, \tag{2.8}$$

and the Moyal bracket of two functions is given by

$$[f,g]_\star(x) = (f\star g)(x) - (g\star f)(x) = 2i\,\sin\left(\frac{1}{2}\,\theta_{ij}\,\partial_{\xi_i}\partial_{\eta_j}\right)f(\xi)g(\eta)\Big|_{\xi=\eta=x}\,. \tag{2.9}$$

The $\star$-commutator (2.9) satisfies the Jacobi identity and the Leibnitz rule. Note that ∂_i defines an inner derivation of the ring $\mathbb{C}[[x_1,\dots,x_d]]/\mathcal{R}_\theta$ but not of the algebra $\mathcal{A}_\theta$.

The action for $\mathrm{U}(N)$ non-commutative Yang-Mills theory is given by

$$S = \frac{1}{4}\int \mathrm{d}^d x\ \ \mathrm{tr}\left(F_{ij}(x)\star F^{ij}(x)\right), \tag{2.10}$$

where

$$F_{ij} = i\,[\nabla_i,\nabla_j]_\star = \partial_i A_j - \partial_j A_i - i\,[A_i,A_j]_\star \tag{2.11}$$

is the non-commutative field strength tensor, and

$$\nabla_i = \partial_i - i\,A_i \tag{2.12}$$

is the gauge connection. The gauge field $A_i(x)$ is a hermitean element of the algebra $\mathbb{M}_N(\mathcal{A}_\theta) = \mathcal{A}_\theta\otimes\mathbb{M}_N(\mathbb{C})$, where $\mathbb{M}_N(\mathbb{C})$ is the elementary C^*-algebra of $N\times N$ matrices and the multiplication in $\mathbb{M}_N(\mathcal{A}_\theta)$ is the tensor product of the $\star$-product (2.4) and ordinary matrix multiplication. The trace in (2.10) is over the matrix indices of the fields. This writing of ordinary Yang-Mills theory on the non-commutative space as the non-commutative gauge theory (2.10) on an ordinary space transmutes the $\mathrm{U}(N)$ colour degrees of freedom into spacetime degrees of freedom along the non-commutative directions. The former gauge theory can be expressed in terms of Weyl operators, as we will describe in the next section.

The action (2.10) is invariant under the infinitesimal non-commutative gauge transformation $A_i\mapsto A_i+\delta_\lambda A_i$, where $\lambda=\lambda^\dagger\in\mathbb{M}_N(\mathcal{A}_\theta)$ and

$$\begin{aligned}\delta_\lambda A_i &= \partial_i\lambda + i\,[\lambda,A_i]_\star\,,\\ \delta_\lambda F_{ij} &= i\,[\lambda,F_{ij}]_\star\,.\end{aligned} \tag{2.13}$$

It is straightforward to show that the commutator of two such transformations with generators $\lambda,\lambda'\in\mathbb{M}_N(\mathcal{A}_\theta)$ is given by

$$[\delta_\lambda,\delta_{\lambda'}]\,A_i = \delta_{i\,[\lambda,\lambda']_\star}A_i\,. \tag{2.14}$$

This implies that the subspace of $\mathbb{M}_N(\mathcal{A}_\theta)$ which generates the non-commutative gauge transforms (2.13) is a Lie algebra with respect to the Moyal bracket. In the following we will present an explicit description of this Lie algebra.

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2.2 Non-commutative gauge symmetry

We will first define more precisely what is meant by a non-commutative gauge symmetry. The algebra $\mathcal{A}_\theta$ can be represented faithfully on a separable Hilbert space $\mathcal{H}$, whose vectors we interpret as fundamental matter fields. The space $\mathcal{H}$ is an $\mathcal{A}_\theta$-module which corresponds to a vector bundle over non-commutative $\mathbb{R}^d$. For example, we can take $\mathcal{H} = \mathcal{H}_{\rm M}$, where

$$\mathcal{H}_{\rm M} = L^2\left(\mathbb{R}^d\,,\,{\rm d}^dx\right) \otimes \mathbb{C}^N\,, \tag{2.15}$$

with the (left) action of $\mathcal{A}_\theta$ defined as

$$f\,:\,\psi\ \longmapsto\ f\star\psi \tag{2.16}$$

for $f \in \mathcal{A}_\theta$ and $\psi \in \mathcal{H}_{\rm M}$. This representation is reducible, but this is not surprising, because the usual representation of the commutative algebra of functions $\mathcal{A}_0$ as operators on Hilbert space is also a highly reducible representation. Namely, we have

$$L^2\left(\mathbb{R}^d,\,{\rm d}^dx\right) = \int_{x\in\mathbb{R}^d}^{\oplus}\delta_x\,, \tag{2.17}$$

where $\delta_x : \mathcal{A}_0 \to \mathbb{C}$ is the evaluation functional at $x \in \mathbb{R}^d$ which is the character of the commutative algebra $\mathcal{A}_0$ given by

$$\delta_x(f) = \int {\rm d}^dy\ \delta(x-y)\,f(y)\,, \qquad f\in\mathcal{A}_0\,, \tag{2.18}$$

and which defines a one-dimensional irreducible representation of $\mathcal{A}_0$ on $L^2(\mathbb{R}^d, {\rm d}^dx)$ via pointwise multiplication as

$$\delta_x(f)\cdot\psi = f(x)\,\psi\,. \tag{2.19}$$

In the non-commutative case, the invariant subspaces are larger and the reducibility of the representation is diminished. To see this explicitly, it is convenient to exploit global euclidean invariance and rotate to a basis of $\mathbb{R}^d$ in which the matrix θ_{ij} assumes its canonical skew-diagonal form with skew-eigenvalues θ_a, $a = 1,\ldots,d/2$. Here and in the following we will assume, for ease of notation, that all θ_a are positive. In this basis the coordinate operators split into $d/2$ mutually commuting blocks in each of which the commutation relation

$$[x_{2a-1}, x_{2a}] = i\,\theta_a\,, \qquad a = 1,\ldots,\frac{d}{2}\,, \tag{2.20}$$

is satisfied. The algebra (2.20) can then be represented by the operators

$$x_{2a} = Q_a\,, \qquad x_{2a-1} = i\,\theta_a\,\frac{\partial}{\partial Q_a}\,, \qquad \text{(no sum on } a) \tag{2.21}$$

acting on the usual quantum mechanical Hilbert space

$$\mathcal{H}_{\rm Q} = \bigotimes_{a=1}^{d/2} L^2\left(\mathbb{R}, {\rm d}Q_a\right) . \tag{2.22}$$

By the Stone-von Neumann theorem, the Schrödinger representation (2.21) and (2.22) is the unique unitary irreducible representation of the Heisenberg algebras (2.20). The space $\mathcal{H}_{\rm Q} \otimes \mathbb{C}^N$ defines a proper, diagonal subspace of the Hilbert space (2.15), thus proving reducibility of the representation (2.16). An invariant subspace can be easily constructed, for example, by introducing the complex coordinates

$$z_a = \frac{1}{\sqrt{2\theta_a}}\left(x_{2a-1} - i\,x_{2a}\right) , \tag{2.23}$$

and the Bargmann space $\mathcal{H}_{\rm B}$ of coherent state wavefunctions

$$\psi_{\rm B}\left(z, z^*\right) = F(z)\ {\rm e}^{-z^\dagger z} , \tag{2.24}$$

where $F(z)$ is an $N\times N$ matrix-valued holomorphic function of $z = (z_a)_{a=1}^{d/2} \in \mathbb{C}^{d/2}$. It is then straightforward to check that $\mathcal{H}_{\rm B}$ is an invariant subspace for the action (2.16) of $\mathcal{A}_\theta$ on the Hilbert space (2.15).

In analogy with the commutative case, we may then consider the gauge transformations

$$\psi \longmapsto \psi^U = U \star \psi , \tag{2.25}$$

where $\psi \in \mathcal{H}_{\rm M}$ and U is a unitary element of the algebra $\mathbb{M}_N(\mathcal{A}_\theta^+)$. Note that since $\mathbb{M}_N(\mathcal{A}_\theta)$ is not a unital algebra, it is necessary to add an identity element to $\mathcal{A}_\theta$ and work with the algebra $\mathcal{A}_\theta^+ = \mathcal{A}_\theta \oplus \mathbb{C}$ in order to properly define unitary elements [26]. Geometrically, this extension corresponds to considering functions on the one-point compactification of $\mathbb{R}^d$.[1] The transformations (2.25) preserve the representation of $\mathcal{A}_\theta$ on $\mathcal{H}_{\rm M}$, and $\star$-unitarity

$$U \star U^\dagger = U^\dagger \star U = \mathbb{1} \tag{2.26}$$

guarantees that they preserve the Hilbert norm of the matter field ψ. The gauge transformation (2.25) is a deformation of that for an ordinary U(N) gauge theory.

The action of the covariant derivative (2.12) as an operator on $\mathcal{H}_{\rm M}$ is defined by

$$\nabla_i(\psi) = \partial_i\psi - i\,A_i \star \psi . \tag{2.27}$$

Since from the definition (2.4) we have

$$\partial_i(f \star \psi) = (\partial_i f) \star \psi + f \star \partial_i \psi , \tag{2.28}$$

[1] Such a unitalization has been used recently in the construction of noncommutative instantons [27].

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it follows that ∇_i satisfies a (left) Leibnitz rule with respect to the representation of $\mathcal{A}_\theta$ on $\mathcal{H}_{\rm M}$,

$$\nabla_i(f\star\psi)=\left(\nabla_i^{\rm ad}f\right)\star\psi+f\star\nabla_i(\psi)\,, \tag{2.29}$$

where the adjoint gauge connection $\nabla_i^{\rm ad}$ is the linear derivation of the algebra $\mathcal{A}_\theta$ defined by

$$\nabla_i^{\rm ad}f=\partial_if+i\,[f,A_i]_\star\,. \tag{2.30}$$

From (2.29) it follows that $\nabla_i(\psi)$ lies in the same representation of $\mathcal{A}_\theta$ as the matter field ψ, and so it should transform in the same way (2.25) as ψ under gauge transformations. This requires the covariant derivatives to transform in the adjoint representation

$$\nabla_i~\longmapsto~\nabla_i^U\,,\qquad\text{with }\nabla_i^U(\psi)=U\star\nabla_i\left(U^\dagger\star\psi\right), \tag{2.31}$$

so that $\nabla_i^U(\psi^U)=\nabla_i(\psi)^U$.

We see therefore that gauge transformations in non-commutative Yang-Mills theory are determined by inner automorphisms $f\mapsto U\star f\star U^\dagger$ of the algebra $\mathbb{M}_N(\mathcal{A}_\theta^+)$. These transformations correspond to rotations of the algebra elements, and they are parametrized by unitary elements U of $\mathbb{M}_N(\mathcal{A}_\theta^+)$ which form the infinite-dimensional group $\mathrm{U}(\mathbb{M}_N(\mathcal{A}_\theta^+))$. Normally, when it is said that a gauge group is finite dimensional, what is really meant is that a gauge transformation is a map from the spacetime manifold into a finite-dimensional Lie group. But the map itself is an element of an infinite-dimensional group, $\mathrm{U}(\mathbb{M}_N(\mathcal{A}_0^+))\cong\mathrm{U}(\mathcal{A}_0^+)\otimes\mathrm{U}(N)$, with $\mathrm{U}(\mathcal{A}_0^+)$ the group of S^1-valued functions. In the non-commutative case the gauge group is itself infinite dimensional, and it is non-abelian even in the simplest instance of $\mathrm{U}(1)$ gauge symmetry. In this paper we will try to understand the Lie algebraic structure related to this group. Unlike the commutative case, however, the group $\mathrm{U}(\mathbb{M}_N(\mathcal{A}_\theta^+))$ is not the tensor product of a function space with a finite-dimensional Lie group, because of the mixing of internal $\mathrm{U}(N)$ and spacetime degrees of freedom that we have alluded to. This will lead to a much richer algebraic structure.

In the infinite-dimensional case, there is no guarantee that we can use the usual construction of a Lie algebra starting from the tangent space to the group at the identity [28]. In fact, it is known [29] that while elements of the form ${\rm e}^{\,i\,\lambda}$, with λ self-adjoint, are unitary, the converse is not generally true. However, we can circumvent these problems in the present case by exploiting the fact that we are really defining the group $\mathrm{U}(\mathbb{M}_N(\mathcal{A}_\theta^+))$ by its representation as operators on a Hilbert space, as in (2.25). Given an $\mathcal{A}_\theta$-module $\mathcal{H}$, let $U\mapsto\mathcal{U}_U$ be a continuous unitary representation of $\mathrm{U}(\mathbb{M}_N(\mathcal{A}_\theta^+))$ on $\mathcal{H}$. Then for each fixed element $U\in\mathrm{U}(\mathbb{M}_N(\mathcal{A}_\theta^+))$, the operators $\mathcal{U}_{t\,U}$, $t\in\mathbb{R}$, form a one-parameter transformation group and we may define an operator on $\mathcal{H}$ by

$$\lambda_U=\lim_{t\to0}\frac{1}{i\,t}\,\left(\mathcal{U}_{t\,U}-\mathbb{1}_{\mathcal{H}}\right). \tag{2.32}$$

By Stone's theorem [30], the infinitesimal operators (2.32) are essentially self-adjoint, and we may write $\mathcal{U}_{t\,U} = \mathrm{e}^{\,i\,t\,\lambda_U}$. In this way it makes sense to speak of the Lie algebra of the group $\mathrm{U}(\mathbb{M}_N(\mathcal{A}_\theta^+))$. Indeed, this is the situation that is anticipated from the relationships between non-commutative gauge theories and matrix models.

We will thereby consider the elements of a basis for the hermitean elements $\mathrm{u}(\mathbb{M}_N(\mathcal{A}_\theta))$ and study the commutation relations between them. In this paper we will limit ourselves to the case of a $\mathrm{U}(1)$ gauge symmetry. The gauge algebra for the cases $N > 1$ can then be obtained from the Lie algebraic tensor product between the algebra we will find and the $\mathrm{u}(N)$ Lie algebra. As we will discuss later on, the gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$ of non-commutative electrodynamics contains all unitary gauge algebras $\mathrm{u}(N)$ in a very precise and exact way, i.e. $\mathrm{U}(1)$ non-commutative Yang-Mills theory contains all possible non-commutative gauge theories with non-abelian unitary gauge groups [22, 31]. This will turn out, in fact, to be a very important geometric feature of the non-commutative gauge group. We will therefore consider $\star$-unitary elements of the form $\mathrm{U}(x) = \mathrm{e}^{\star\,i\,\lambda(x)}$, where $\lambda(x)$ is a real-valued function on $\mathbb{R}^d$ and the $\star$-exponential is defined by the understanding that all products in its Taylor series representation are $\star$-products. The infinitesimal gauge transform of the covariant derivative is then given by

$$\nabla_i(\psi) \;\longmapsto\; \nabla_i(\psi) + i\,\lambda \star \nabla_i(\psi) - i\,\nabla_i(\lambda \star \psi)\,, \tag{2.33}$$

which from (2.27) immediately implies the gauge transformation rule (2.13). Generically then, the gauge algebra acts on elements of the algebra $\mathcal{A}_\theta$ via the homogeneous, covariant transformations

$$f \;\longmapsto\; f + i\,[\lambda, f]_\star\,. \tag{2.34}$$

There is a very important property of the algebra of non-commutative $\mathbb{R}^d$ that we will exploit in the following. Unlike the case of ordinary $\mathbb{R}^d$, where the algebra $\mathcal{A}_0$ of functions is commutative and there are no inner automorphisms, the gauge symmetries here act via rotations of functions and correspond to internal fluctuations of the spacetime geometry in the above sense. The inner automorphisms form a normal subgroup $\mathrm{Inn}(\mathcal{A}_\theta)$ of the automorphism group $\mathrm{Aut}(\mathcal{A}_\theta)$ of the algebra $\mathcal{A}_\theta$. The exact sequence of groups

$$\mathbb{1} \;\longrightarrow \mathrm{Inn}\,(\mathcal{A}_\theta) \longrightarrow \mathrm{Aut}\,(\mathcal{A}_\theta) \longrightarrow \mathrm{Out}\,(\mathcal{A}_\theta) \longrightarrow\; \mathbb{1} \tag{2.35}$$

defines the remaining outer automorphisms of $\mathcal{A}_\theta$ such that the full automorphism group $\mathrm{Aut}(\mathcal{A}_\theta)$ is the semi-direct product of $\mathrm{Inn}(\mathcal{A}_\theta)$ by the natural action of $\mathrm{Out}(\mathcal{A}_\theta)$. For commutative $\mathbb{R}^d$, there are only outer automorphisms,[2] and the group $\mathrm{Aut}(\mathcal{A}_0)$ is naturally isomorphic to the group of diffeomorphisms of $\mathbb{R}^d$ [32]. Given

[2] More precisely, the inner automorphisms generate ordinary gauge transformations.

a smooth function $\phi : \mathbb{R}^d \to \mathbb{R}^d$, there is a natural automorphism $\alpha_\phi : \mathcal{A}_0 \to \mathcal{A}_0$ defined by

$$\alpha_\phi(f) = f \circ \phi^{-1} , \qquad f \in \mathcal{A}_0 . \tag{2.36}$$

Like the inner automorphisms in the non-commutative case, the outer automorphisms (2.36) can be represented via unitary conjugation when the algebra $\mathcal{A}_0$ is represented by operators on the Hilbert space (2.15). Given a diffeomorphism ϕ of $\mathbb{R}^d$, we may define a unitary operator U_ϕ on $\mathcal{H}_{\rm M}$ by

$$U_\phi\, \psi(x) = \left|\frac{\partial\phi}{\partial x}\right|^{1/2} \psi\left(\phi^{-1} x\right) , \qquad \psi \in \mathcal{H}_{\rm M} . \tag{2.37}$$

In this way, we may identify the automorphism group of the algebra $\mathcal{A}_0$ with the group $\mathrm{U}(\mathcal{H}_{\rm M})$ of unitary endomorphisms of the Hilbert space $\mathcal{H}_{\rm M}$ (or more precisely, as we will see in section 4.2, with the projective subgroup of this unitary group).

In the non-commutative case we will find that some of these outer automorphisms are deformed into inner automorphisms and thereby generate gauge transformations. This will follow from the fact that, generally, the derivation ∂_i generates an infinitesimal automorphism of the algebra. For the algebra (2.1) it determines the inner automorphism (2.7) of the ring $\mathbb{C}[[x_1, \ldots, x_d]]/\mathcal{R}_\theta$. In fact, since the algebra has a trivial center, any linear derivation of $\mathcal{A}_\theta$ can be realized as an inner automorphism of $\mathbb{C}[[x_1, \ldots, x_d]]/\mathcal{R}_\theta$. This is a property particular to non-commutative $\mathbb{R}^d$, because in that case the coordinates x_i generate the algebra of functions. It does not hold, for example, on the non-commutative torus. This feature will enable us to use the mixing of gauge and spacetime degrees of freedom to realize certain geometric transformations of $\mathbb{R}^d$ as genuine gauge symmetries of non-commutative Yang-Mills theory. In this way we will see in fact that the automorphism group of the non-commutative algebra $\mathcal{A}_\theta$ lies in between the two extremes generated by the commutative algebra $\mathcal{A}_0$, for which $\mathrm{Aut}(\mathcal{A}_0) = \mathrm{Out}(\mathcal{A}_0)$, and a finite-dimensional matrix algebra $\mathbb{M}_N(\mathbb{C})$ for which all automorphisms are inner automorphisms.

3. Explicit presentations

In this section we will construct two explicit representations of the Lie algebra $\mathrm{u}(\mathcal{A}_\theta)$, each of which will be useful in its own right in the following. The first one will be important for the representation $\mathcal{H}_{\rm M}$ of the algebra $\mathcal{A}_\theta$ in terms of fundamental matter fields, while the second one will be pertinent to the Schrödinger $\mathcal{A}_\theta$-module $\mathcal{H}_{\rm Q}$ which will be used later on to explicitly identify the non-commutative gauge algebra.

3.1 Symmetric representation

Using the formulas $x_i^{\star n^i} = x_i^{n^i}$ and $x_i \star x_j = x_i x_j + \frac{i}{2}\,\theta_{ij}$, it is possible to express the real-valued gauge function $\lambda(x)$ as a series in $\star$-monomials generated by the x_i's (with the appropriate convergence criterion on the expansion coefficients). However, although the x_i are hermitean operators, the noncommutativity of the coordinates implies in general that $x_i^{n^i} \star x_j^{n^j} \neq x_j^{n^j} \star x_i^{n^i}$ for $i \neq j$, and so the $\star$-monomials do not constitute a good basis for the space of hermitean operators. We need to use appropriate hermitean combinations of the $\star$-monomials in the expansions of the elements of the algebra. We will choose as basis

$$T_{\vec{n}}(x) = {\circ\atop\circ}\, x_1^{\star n^1} \star \cdots \star x_d^{\star n^d} \,{\circ\atop\circ}\,, \tag{3.1}$$

where $\vec{n} = (n^1, \ldots, n^d)$ is a d-dimensional vector of non-negative integers, and we have defined the symmetric $\star$-product of functions by

$${\circ\atop\circ}\, f_1 \star \cdots \star f_m \,{\circ\atop\circ} = \frac{1}{m!} \sum_{\sigma \in S_m} f_{\sigma(1)} \star \cdots \star f_{\sigma(m)}\,. \tag{3.2}$$

In other words, the operator $T_{\vec{n}}$ is given by a sum over all permutations $\sigma \in S_{|\vec{n}|}$, $|\vec{n}| \equiv \sum_i n^i$, of $\star$-products of the coordinates such that x_i appears exactly n^i times in each $\star$-monomial. In the following we will define the $\star$-symmetrization of a functional $\mathcal{F}(f_1, \ldots, f_m)$ of m functions $f_1, \ldots, f_m$ by first formally expanding $\mathcal{F}$ as a Taylor series and then applying the symmetrization operation to each monomial,

$${\circ\atop\circ}\, \mathcal{F}\left(f_1, \ldots, f_m\right) {\circ\atop\circ} = \sum_{\vec{k} \in \mathbb{Z}_+^m} \frac{\mathcal{F}^{(k^1, \ldots, k^m)}(0, \ldots, 0)}{k^1! \cdots k^m!}\; {\circ\atop\circ}\, f_1^{k^1} \star \cdots \star f_m^{k^m} \,{\circ\atop\circ}\,. \tag{3.3}$$

The operators (3.1) are hermitean and constitute an improper basis of the algebra $\mathrm{u}(\mathcal{A}_\theta)$, in fact they span the ring $\mathbb{C}[[x_1, \ldots, x_d]]/\mathcal{R}_\theta$. With the use of the commutation relations (2.1), one can easily see that (3.1) coincides with the ordinary, undeformed monomial product. Thus a generic Schwartz function $\lambda(x)$ on $\mathbb{R}^d$ can be expanded as

$$\lambda = \sum_{\vec{n} \in \mathbb{Z}_+^d} \frac{\partial_1^{n^1} \cdots \partial_d^{n^d} \lambda(0)}{n^1! \cdots n^d!}\, T_{\vec{n}}\,. \tag{3.4}$$

This determines the gauge functions of $\mathrm{u}(\mathcal{A}_\theta)$ in terms of a basis of the vector space spanned by homogeneous symmetric polynomials in the generators of the Lie algebra generated by the x_i's. In other words, elements of $\mathrm{u}(\mathcal{A}_\theta)$ lie in the enveloping algebra of the Lie algebra (2.1). In the following we will compute the Moyal brackets of the elements (3.1),

$$\left[T_{\vec{n}}\,,\, T_{\vec{m}}\right]_\star = \sum_{\vec{p} \in \mathbb{Z}_+^d} c_{\vec{n}\vec{m}\vec{p}}\; T_{\vec{p}}\,, \tag{3.5}$$

where $c_{\vec{n}\vec{m}\vec{p}}$ are the structure constants of the Lie algebra $\mathrm{u}(\mathcal{A}_\theta)$ in the basis (3.1).

The calculation of the $\star$-commutators (3.5) is most efficiently done by exploiting the one-to-one correspondence between non-local, non-commutative fields on a commutative space and local, commutative fields on a non-commutative space. This is achieved via the Weyl quantization map [33], which associates to every function $f(x)$ on $\mathbb{R}^d \to \mathbb{C}$ an operator-valued function of self-adjoint operators $\hat{x}_i$ which generate the algebra $[\hat{x}_i, \hat{x}_j] = i\,\theta_{ij}$, and which act faithfully on a particular Hilbert space $\mathcal{H}$. Elements of the algebra $\mathcal{A}_\theta$ are then represented as operators on $\mathcal{H}$, i.e. as elements of the endomorphism algebra $\mathrm{End}(\mathcal{H})$ of the Hilbert space. We will denote this representation by $\mathcal{A}_\theta(\mathcal{H})$. The Weyl map $\Omega : \mathcal{A}_0 \to \mathcal{A}_\theta(\mathcal{H})$ is given by

$$F(\hat{x}) = \Omega(f) = \int \mathrm{d}^d x\ f(x) \int \frac{\mathrm{d}^d \xi}{(2\pi)^d}\ \mathrm{e}^{\,i\,\xi\cdot(\hat{x}-x)}\,, \tag{3.6}$$

and it generalizes the transformation which is usually used in quantum mechanics to associate a quantum operator to a function on a classical phase space. The Weyl transform has an inverse, known as the Wigner map [34], which is given by

$$f(x) = \Omega^{-1}(F) = \pi^{d/2}\ \mathrm{Pfaff}(\theta) \int \frac{\mathrm{d}^d \xi}{(2\pi)^d}\ \mathrm{Tr}_{\mathcal{H}}\left(F(\hat{x})\ \mathrm{e}^{\,i\,\xi\cdot(\hat{x}-x)}\right), \tag{3.7}$$

where the trace over states of $\mathcal{H}$ is equivalent to integration over the non-commuting coordinates $\hat{x}_i$. The useful property of the Weyl and Wigner maps is that they generate an isomorphism $\mathcal{A}_\theta \leftrightarrow \mathcal{A}_\theta(\mathcal{H})$, i.e.

$$\begin{aligned} \Omega(f \star g) &= \Omega(f)\,\Omega(g)\,, \\ \Omega^{-1}(F\,G) &= \Omega^{-1}(F) \star \Omega^{-1}(G)\,. \end{aligned} \tag{3.8}$$

Moreover, spacetime averages of fields map to traces of Weyl operators,

$$\int \mathrm{d}^d x\ f(x) = \pi^{d/2}\ \mathrm{Pfaff}(\theta)\ \mathrm{Tr}_{\mathcal{H}}\left(\Omega(f)\right), \tag{3.9}$$

and under the Weyl-Wigner correspondence the U(1) non-commutative gauge theory (2.10) becomes ordinary Yang-Mills theory on the non-commutative space,

$$S = \frac{\pi^{d/2}\,\mathrm{Pfaff}(\theta)}{4}\ \mathrm{Tr}_{\mathcal{H}}\left(\Omega(F_{ij})\,\Omega(F^{ij})\right). \tag{3.10}$$

In this way we may regard non-commutative gauge theory as ordinary Yang-Mills theory with the extended, infinite-dimensional local gauge symmetry algebra $\mathrm{u}(\mathcal{A}_\theta)$. Note that from (3.8) it follows directly that the $\star$-product possesses the same algebraic properties as the ordinary operator product in $\mathrm{End}(\mathcal{H})$, i.e. it is associative but non commutative, while (3.9) shows explicitly that spacetime integrals of $\star$-products of functions have precisely the same cyclic permutation symmetries as traces of operator products (c.f. (2.5)).

In the present case, the real advantage of using the Weyl-Wigner correspondence is the operator ordering that is provided by the map (3.6). By definition, it symmetrically orders operator products. This implies that if we define the Weyl operators

$$\hat{T}_{\vec{n}} \equiv \Omega\left(x_1^{n^1} \cdots x_d^{n^d}\right), \tag{3.11}$$

then $\hat{T}_{\vec{n}} = \Omega(T_{\vec{n}})$ and the Lie algebra (3.5) can be computed as

$$\left[\hat{T}_{\vec{n}}\,,\,\hat{T}_{\vec{m}}\right] = \sum_{\vec{p}\in\mathbb{Z}^d_+} c_{\vec{n}\vec{m}\vec{p}}\,\hat{T}_{\vec{p}}\,. \tag{3.12}$$

A further simplification comes from rotating to a basis in which the commutation relations of the non-commutative space assume the skew-block form (2.20). By denoting $\hat{T}^{(a)}_{\vec{n}} = \Omega(x_{2a-1}^{n^{2a-1}}\,x_{2a}^{n^{2a}})$, the left-hand side of (3.12) may then be computed from

$$\left[\hat{T}_{\vec{n}}\,,\,\hat{T}_{\vec{m}}\right] = \sum_{a=1}^{d/2} \hat{T}^{(1)}_{\vec{n}+\vec{m}}\cdots\hat{T}^{(a-1)}_{\vec{n}+\vec{m}}\left[\hat{T}^{(a)}_{\vec{n}}\,,\,\hat{T}^{(a)}_{\vec{m}}\right]\hat{T}^{(a+1)}_{\vec{n}+\vec{m}}\cdots\hat{T}^{(d)}_{\vec{n}+\vec{m}}\,, \tag{3.13}$$

and the commutators appearing on the right-hand side of (3.13) can be calculated as

$$\left[\hat{T}^{(a)}_{\vec{n}},\hat{T}^{(a)}_{\vec{m}}\right] = \Omega\left((x_{2a-1}^{n^{2a-1}}\,x_{2a}^{n^{2a}})\star(x_{2a-1}^{m^{2a-1}}\,x_{2a}^{m^{2a}}) - (x_{2a-1}^{m^{2a-1}}\,x_{2a}^{m^{2a}})\star(x_{2a-1}^{n^{2a-1}}\,x_{2a}^{n^{2a}})\right)\,. \tag{3.14}$$

We are therefore left with a simple calculation which involves only the $\star$-products of monomials, and not the combinatorics involved in the symmetrization operation.

The computation of (3.14) is straightforward. For this, we note that in a given skew-block a, the $\star$-product (2.4) may be written as

$$\begin{aligned}(f\star g)^{(a)} &= \sum_{r=0}^\infty\sum_{s=0}^\infty \frac{(-1)^r\,(i\,\theta_a)^{r+s}}{2^{r+s}\,r!\,s!}\,\frac{\partial^{r+s}f}{\partial x_{2a-1}^r\partial x_{2a}^s}\,\frac{\partial^{r+s}g}{\partial x_{2a-1}^s\partial x_{2a}^r}\\ &= \sum_{n=0}^\infty\left(\frac{i\,\theta_a}{2}\right)^n\sum_{r=0}^n\frac{(-1)^r}{(n-r)!\,r!}\left(\partial_{2a-1}^r\partial_{2a}^{n-r}f\right)\left(\partial_{2a-1}^{n-r}\partial_{2a}^r g\right)\,,\end{aligned} \tag{3.15}$$

from which we may express the Moyal bracket (2.9) as

$$\begin{aligned}[f,g]^{(a)}_\star &= 2i\sum_{p=0}^\infty\frac{(\theta_a)^{2p+1}}{4^p}\times\\ &\quad\times\sum_{l=0}^{2p+1}\frac{(-1)^{p+l}}{l!\,(2p+1-l)!}\left(\partial_{2a-1}^l\partial_{2a}^{2p+1-l}f\right)\left(\partial_{2a-1}^{2p+1-l}\partial_{2a}^l g\right)\,.\end{aligned} \tag{3.16}$$

Setting $f = x_{2a-1}^{n^{2a-1}}\,x_{2a}^{n^{2a}}$ and $g = x_{2a-1}^{m^{2a-1}}\,x_{2a}^{m^{2a}}$ in (3.16) thereby leads to

$$\begin{aligned}\left[T^{(a)}_{\vec{n}}\,,\,T^{(a)}_{\vec{m}}\right]_\star &= 2i\sum_{0\le p\le\frac12(\min(n^{2a-1},m^{2a})+\min(n^{2a},m^{2a-1})-1)}\frac{(\theta_a)^{2p+1}}{4^p}\times\\ &\times\sum_{l=2p+1-\min(n^{2a},m^{2a-1},2p+1)}^{\min(n^{2a-1},m^{2a},2p+1)}\frac{(-1)^{p+l}}{l!\,(2p+1-l)!}\,\frac{n^{2a-1}!\,n^{2a}!}{(n^{2a-1}-l)!\,(n^{2a}-2p-1+l)!}\times\\ &\times\frac{m^{2a-1}!\,m^{2a}!}{(m^{2a-1}-2p-1+l)!\,(m^{2a}-l)!}\,T^{(a)}_{\vec{n}+\vec{m}-\underline{(2p+1)}}\,,\end{aligned} \tag{3.17}$$

where generally $\underline{r} = (r,r,\dots,r)$ denotes the integer vector whose components are all equal to $r\in\mathbb{Z}_+$.

We can now rotate back to general form by using (3.13), and write a commutation relation of the form (3.5). For the structure constants of the full Lie algebra we then arrive at

$$\begin{aligned} c_{\vec{n}\vec{m}(\vec{n}+\vec{m}-(2p+1)(\vec{e}_i+\vec{e}_j))} &= 2i\,\frac{(\theta_{ij})^{2p+1}}{4^p}\sum_{l=2p+1-\min(n^j,m^i,2p+1)}^{\min(n^i,m^j,2p+1)}\frac{(-1)^{p+l}}{l!\,(2p+1-l)!}\times \\ &\quad\times\frac{n^i!\,n^j!}{(n^i-l)!\,(n^j-2p-1+l)!}\,\frac{m^i!\,m^j!}{(m^i-2p-1+l)!\,(m^j-l)!} \\ c_{\vec{n}\vec{m}\vec{k}} &= 0 \qquad \text{for} \qquad \vec{k}\neq\vec{n}+\vec{m}-(2p+1)\,(\vec{e}_i+\vec{e}_j)\,, \\ 2p+1 &> \max_{i<j}\left\{\min(n^i,m^j)+\min(n^j,m^i)\right\}, \end{aligned} \tag{3.18}$$

where $\vec{e}_i$ is the standard basis of the hypercubic lattice $\mathbb{Z}_+^d$,

$$(\vec{e}_i)^j = \delta_i^j\,. \tag{3.19}$$

Although the expressions (3.17) and (3.18) as they stand are not particularly transparent, we will see in the following how their explicit expansions in θ_{ij} reveal some remarkable geometrical features of the gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$.

3.2 Density matrix representation

Another very important representation of the Lie algebra $\mathrm{u}(\mathcal{A}_\theta)$ arises on the space of Weyl operators $\mathcal{A}_\theta(\mathcal{H})$ when $\mathcal{H}$ is taken to be the Hilbert space (2.22) of quantum mechanics (with respect to the skew-diagonalization of θ_{ij}), i.e. the Schrödinger representation of the Heisenberg commutation relations (2.20). Let us fix, as before, an integer vector $\vec{n}\in\mathbb{Z}_+^d$, and for each $a=1,\dots,d/2$ consider the Weyl operators $\hat{z}_a$ corresponding to the complex coordinates (2.23). These operators obey the commutation relations

$$\left[\hat{z}_a\,,\,\hat{z}_a^\dagger\right]=1 \tag{3.20}$$

in each commuting skew-block, and the Weyl transform (3.6) may be expressed in terms of them via the substitutions

$$\hat{x}_{2a-1}=\sqrt{\frac{\theta_a}{2}}\,\left(\hat{z}_a+\hat{z}_a^\dagger\right), \qquad \hat{x}_{2a}=i\,\sqrt{\frac{\theta_a}{2}}\,\left(\hat{z}_a-\hat{z}_a^\dagger\right). \tag{3.21}$$

The Hilbert space (2.22) may then be represented in terms of the standard Fock space of creation and annihilation operators as

$$L^2(\mathbb{R},\mathrm{d}Q_a)=\ell^2(\mathbb{Z}_+)=\bigoplus_{n^{2a}=0}^{\infty}\mathbb{C}|n^{2a}\rangle\,, \tag{3.22}$$

where $|n^{2a}\rangle$ are the orthonormal eigenstates of the number operator $\hat{z}_a^\dagger\,\hat{z}_a$ with eigenvalues $n^{2a}\in\mathbb{Z}_+$, and the action of the Weyl operators is defined by

$$\hat{z}_a|n^{2a}\rangle=\sqrt{n^{2a}}\,|n^{2a}-1\rangle\,, \qquad \hat{z}_a^\dagger|n^{2a}\rangle=\sqrt{n^{2a}+1}\,|n^{2a}+1\rangle\,. \tag{3.23}$$

A basis for the Lie algebra $\mathrm{u}(\mathcal{A}_\theta(\mathcal{H}_{\rm Q}))$ of hermitean Weyl operators on the Hilbert space (2.22) is given by the basis for density matrices,

$$\hat\Sigma_{\vec n}^{(\epsilon_1\cdots\epsilon_{d/2})}=\bigotimes_{a=1}^{d/2}\hat\Sigma_{\vec n}^{(a)\epsilon_a}\,,\qquad\epsilon_a=\pm\,,\tag{3.24}$$

where

$$\begin{aligned}\hat\Sigma_{\vec n}^{(a)+}&=i\,|n^{2a}\rangle\langle n^{2a-1}|-i\,|n^{2a-1}\rangle\langle n^{2a}|\,,\\ \hat\Sigma_{\vec n}^{(a)-}&=|n^{2a}\rangle\langle n^{2a-1}|+|n^{2a-1}\rangle\langle n^{2a}|\end{aligned}\tag{3.25}$$

for each $a=1,\dots,d/2$ span the space of self-adjoint operators on the Fock space (3.22). To compute the Wigner functions (3.7) corresponding to the Weyl operators (3.24), we use the standard Groenewold distribution functions for the energy eigenstates of the one-dimensional harmonic oscillator [24, 35]

$$\begin{aligned}E_{n^{2a-1}n^{2a}}(z_a,z_a^*)&\equiv\Omega^{-1}\left(|n^{2a-1}\rangle\langle n^{2a}|\right)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}{\rm d}\xi_{2a}\,{\rm e}^{\,-i\xi_{2a}x_{2a}}\left\langle x_{2a-1}-\frac{\theta_a}{2}\xi_{2a}|n^{2a-1}\right\rangle\times\\&\quad\times\left\langle n^{2a}|x_{2a-1}+\frac{\theta_a}{2}\xi_{2a}\right\rangle\\&=\frac{(-1)^{n^{2a}}}{\pi}\sqrt{\frac{n^{2a}!}{n^{2a-1}!}}\,\left(4\theta_a|z_a|^2\right)^{\frac{n^{2a-1}-n^{2a}}{2}}\ {\rm e}^{\,-2\theta_a|z_a|^2}\times\\&\quad\times{\rm e}^{\,i(n^{2a-1}-n^{2a})\arg(z_a)}\,L_{n^{2a}}^{n^{2a-1}-n^{2a}}\left(4\theta_a|z_a|^2\right),\end{aligned}\tag{3.26}$$

where z_a are the complex coordinates (2.23) and

$$L_n^\beta(t)=\frac{t^{-\beta}\ {\rm e}^{\,t}}{n!}\,\frac{d^n}{dt^n}\,\left(t^{\beta+n}\ {\rm e}^{\,-t}\right)\tag{3.27}$$

are the associated Laguerre functions. Since $E_{n^{2a-1}n^{2a}}=E^*_{n^{2a}n^{2a-1}}$, the Wigner functions which generate the gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$ in the occupation number basis of the Schrödinger representation are thereby given as

$$\begin{aligned}\Sigma_{\vec n}^{(\epsilon_1\cdots\epsilon_{d/2})}(x)&=\prod_{a=1}^{d/2}\Sigma_{\vec n}^{(a)\epsilon_a}(z_a,z_a^*)\,,\qquad\epsilon_a=\pm\,,\\ \Sigma_{\vec n}^{(a)+}(z_a,z_a^*)&=-\frac{2(-1)^{n^{2a}}}{\pi}\sqrt{\frac{n^{2a}!}{n^{2a-1}!}}\,\left(4\theta_a|z_a|^2\right)^{\frac{n^{2a-1}-n^{2a}}{2}}\ {\rm e}^{\,-2\theta_a|z_a|^2}\times\\&\quad\times\sin\left(\left(n^{2a-1}-n^{2a}\right)\,\arg(z_a)\right)\,L_{n^{2a}}^{n^{2a-1}-n^{2a}}\left(4\theta_a|z_a|^2\right),\\ \Sigma_{\vec n}^{(a)-}(z_a,z_a^*)&=\frac{2(-1)^{n^{2a}}}{\pi}\sqrt{\frac{n^{2a}!}{n^{2a-1}!}}\,\left(4\theta_a|z_a|^2\right)^{\frac{n^{2a-1}-n^{2a}}{2}}\ {\rm e}^{\,-2\theta_a|z_a|^2}\times\\&\quad\times\cos\left(\left(n^{2a-1}-n^{2a}\right)\,\arg(z_a)\right)\,L_{n^{2a}}^{n^{2a-1}-n^{2a}}\left(4\theta_a|z_a|^2\right).\end{aligned}\tag{3.28}$$

The commutation relations of the operators (3.25) can be easily worked out using the orthonormality relations $\langle m|n\rangle = \delta_{nm}$, and for the Moyal brackets of the functions (3.28) we thus find

$$
\begin{aligned}
\left[\Sigma_{\vec{n}}^{(a)+}\,,\,\Sigma_{\vec{m}}^{(a)+}\right]_\star &= -i\left(\delta_{n^{2a-1}m^{2a-1}}\,\Sigma^{(a)+}_{n^{2a}m^{2a}} + \delta_{n^{2a}m^{2a}}\,\Sigma^{(a)+}_{n^{2a-1}m^{2a-1}} - \right.\\
&\qquad\left. - \,\delta_{n^{2a-1}m^{2a}}\,\Sigma^{(a)+}_{n^{2a}m^{2a-1}} - \delta_{n^{2a}m^{2a-1}}\,\Sigma^{(a)+}_{n^{2a-1}m^{2a}}\right),\\
\left[\Sigma_{\vec{n}}^{(a)+}\,,\,\Sigma_{\vec{m}}^{(a)-}\right]_\star &= i\left(\delta_{n^{2a-1}m^{2a-1}}\,\Sigma^{(a)-}_{n^{2a}m^{2a}} - \delta_{n^{2a}m^{2a}}\,\Sigma^{(a)-}_{n^{2a-1}m^{2a-1}} + \right.\\
&\qquad\left. + \,\delta_{n^{2a-1}m^{2a}}\,\Sigma^{(a)-}_{n^{2a}m^{2a-1}} - \delta_{n^{2a}m^{2a-1}}\,\Sigma^{(a)-}_{n^{2a-1}m^{2a}}\right),\\
\left[\Sigma_{\vec{n}}^{(a)-}\,,\,\Sigma_{\vec{m}}^{(a)-}\right]_\star &= -i\left(\delta_{n^{2a-1}m^{2a-1}}\,\Sigma^{(a)+}_{n^{2a}m^{2a}} + \delta_{n^{2a}m^{2a}}\,\Sigma^{(a)+}_{n^{2a-1}m^{2a-1}} + \right.\\
&\qquad\left. + \,\delta_{n^{2a-1}m^{2a}}\,\Sigma^{(a)+}_{n^{2a}m^{2a-1}} + \delta_{n^{2a}m^{2a-1}}\,\Sigma^{(a)+}_{n^{2a-1}m^{2a}}\right). \qquad (3.29)
\end{aligned}
$$

It is intriguing to note that the gauge algebra in this representation also canonically has the structure of a Lie superalgebra. We can define a $\mathbb{Z}_2$-grading on $\mathrm{u}(\mathcal{A}_\theta)$ by

$$
\deg\,\Sigma_{\vec{n}}^{(a)\epsilon_a} = \frac{1-\epsilon_a}{2}\,, \qquad (3.30)
$$

and compute the $\star$-anticommutators of the odd functions $\Sigma_{\vec{n}}^{(a)-}$ to get

$$
\begin{aligned}
\left\{\Sigma_{\vec{n}}^{(a)-}\,,\,\Sigma_{\vec{m}}^{(a)-}\right\}_\star &= \delta_{n^{2a-1}m^{2a-1}}\,\Sigma^{(a)-}_{n^{2a}m^{2a}} + \delta_{n^{2a}m^{2a}}\,\Sigma^{(a)-}_{n^{2a-1}m^{2a-1}}\\
&\qquad + \delta_{n^{2a-1}m^{2a}}\,\Sigma^{(a)-}_{n^{2a}m^{2a-1}} + \delta_{n^{2a}m^{2a-1}}\,\Sigma^{(a)-}_{n^{2a-1}m^{2a}}\,, \qquad (3.31)
\end{aligned}
$$

where

$$
\{f,g\}_\star(x) = (f\star g)(x) + (g\star f)(x) = 2\cos\left(\frac{1}{2}\,\theta_{ij}\,\partial_{\xi_i}\partial_{\eta_j}\right)\,f(\xi)g(\eta)\Big|_{\xi=\eta=x}\,. \qquad (3.32)
$$

It would be interesting to interpret this structure as some sort of "hidden" supersymmetry of non-commutative Yang-Mills theory. In any case, with the definition (3.30), the Lie algebra (3.29) naturally possesses a multiplicative $\mathbb{Z}_2$-grading.

3.3 Relations between the presentations

The advantage of the occupation number basis over the basis of $\star$-monomials is the simplicity of the structure constants in (3.29). On the other hand, the Moyal brackets of the symmetric representation can be written down succinctly in an arbitrary choice of axes for $\mathbb{R}^d$, contrary to the representation of the previous subsection. Notice also that the dependence on the noncommutativity parameters in the density matrix case is completely absorbed into the functions (3.28). This feature makes it difficult to analyse the algebra as a deformation of that for an ordinary gauge theory. In particular, the functions do not go over smoothly into a basis of functions for $\mathcal{A}_0$

in the commutative limit $\theta_{ij} \to 0$. In contrast, the structure constants (3.18) are amenable to explicit analysis order by order in the deformation parameters. There are some interesting relationships between the two bases we have constructed that we shall now proceed to analyse.

Let us first explicitly describe, for the sake of completeness, the transformation between the two presentations of the gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$ given in this section, i.e. the change of basis between the two sets of functions (3.1) and (3.28). It suffices to do this in each commuting skew-block a. Thus we want to express the functions $\Sigma_{\vec{n}}^{(a)\pm} = \Omega^{-1}(\hat{\Sigma}_{\vec{n}}^{(a)\pm})$ in terms of $T_{\vec{n}}^{(a)}$. Again it is useful to notice that $\Omega^{-1}(\hat{T}_{\vec{n}}^{(a)}) = x_{2a-1}^{n^{2a-1}} x_{2a}^{n^{2a}}$. Furthermore, from (3.25) it follows that $\Sigma^{(a)\pm}_{n^{2a-1}n^{2a}} = \mp\,\Sigma^{(a)\pm}_{n^{2a}n^{2a-1}}$. Owing to this latter property we can always assume that $n^{2a-1} \geq n^{2a}$. The desired change of basis is then given simply by the Taylor series expansions of the analytic functions $\Sigma_{\vec{n}}^{(a)\pm}(x)$ in (3.28).

For this, we first rewrite the energy distribution functions (3.26) as

$$\begin{aligned} E_{n^{2a-1}n^{2a}} = {} & \frac{(-1)^{n^{2a}}}{\pi}\sqrt{\frac{n^{2a}!}{n^{2a-1}!}}\,(4\theta_a)^{\frac{n^{2a-1}-n^{2a}}{2}}\,(x_{2a-1}+i\,x_{2a})^{n^{2a-1}-n^{2a}} \times \\ & \times\ \mathrm{e}^{-2\theta_a(x_{2a-1}^2+x_{2a}^2)}L_{n^{2a}}^{n^{2a-1}-n^{2a}}\left(4\theta_a(x_{2a-1}^2+x_{2a}^2)\right). \end{aligned} \tag{3.33}$$

This expression can be expanded in a Taylor series by using the identity [36]

$$L_n^\beta(t+s) = \mathrm{e}^{\,s}\sum_{m=0}^\infty \frac{(-1)^m}{m!}\,s^m\,L_n^{\beta+m}(t) \tag{3.34}$$

and the explicit expression for the associated Laguerre polynomials

$$L_n^k(t) = \sum_{m=0}^n (-1)^m \begin{pmatrix} n+k \\ n-m \end{pmatrix} \frac{t^m}{m!}\,. \tag{3.35}$$

By using in addition the binomial theorem and the Taylor series expansion of the exponential function we can expand (3.33) as

$$\begin{aligned} E_{n^{2a-1}n^{2a}} = {} & \frac{(-1)^{n^{2a}}}{\pi}\sqrt{\frac{n^{2a}!}{n^{2a-1}!}}\,(4\theta_a)^{\frac{n^{2a-1}-n^{2a}}{2}} \sum_{k^{2a-1},k^{2a}=0}^\infty \frac{(-2\theta_a x_{2a-1}^2)^{k^{2a-1}}}{k^{2a-1}!}\,\frac{(2\theta_a x_{2a}^2)^{k^{2a}}}{k^{2a}!} \times \\ & \times \sum_{k=0}^\infty \frac{(-4\theta_a\,x_{2a}^2)^k}{k!}\sum_{l=0}^{n^{2a}} \begin{pmatrix} n^{2a-1}+k \\ n^{2a}-l \end{pmatrix} \frac{(-4\theta_a\,x_{2a-1}^2)^l}{l!} \times \\ & \times \sum_{p=0}^{n^{2a-1}-n^{2a}} i^p \begin{pmatrix} n^{2a-1}-n^{2a} \\ p \end{pmatrix} x_{2a-1}^{n^{2a-1}-n^{2a}-p}\,x_{2a}^p\,. \end{aligned} \tag{3.36}$$

Collecting the powers of x_{2a-1} and x_{2a}, and taking the imaginary and real parts of (3.36) leads finally to the change of basis from the $\star$-monomials $T_{\vec{n}}$ to the Wigner functions $\Sigma_{\vec{n}}^{(\epsilon_1\cdots\epsilon_{d/2})}$ of the density matrices,

$$\Sigma_{\vec{n}}^{(a)\epsilon_a} = -\frac{2^{\frac{n^{2a-1}-n^{2a}}{2}+1}\,\epsilon_a}{\pi}\,\sqrt{\frac{n^{2a}!}{n^{2a-1}!}}\times$$
$$\times{\sum_{\vec{m}}}'\sum_{k=0}^{\infty}\sum_{l=0}^{n^{2a}}\sum_{p=0}^{\left[\frac{n^{2a-1}-n^{2a}-\frac{1-\epsilon_a}{2}}{2}\right]}(-1)^{\frac{m^{2a-1}-n^{2a-1}+2p+\frac{1+\epsilon_a}{2}}{2}}\times$$
$$\times\frac{2^{k+l}\,\left(\sqrt{2\theta_a}\right)^{m^{2a-1}+m^{2a}}}{k!\,l!\,\left(\frac{m^{2a-1}+n^{2a}-n^{2a-1}+2p+\frac{1+\epsilon_a}{2}}{2}-l\right)!\,\left(\frac{m^{2a}-2p-\frac{1+\epsilon_a}{2}}{2}-k\right)!}\times$$
$$\times\begin{pmatrix}n^{2a-1}+k\\ n^{2a}-l\end{pmatrix}\begin{pmatrix}n^{2a-1}-n^{2a}\\ 2p+\frac{1+\epsilon_a}{2}\end{pmatrix}\,T_{\vec{m}}^{(a)}\,. \tag{3.37}$$

The prime on the first sum in (3.37) means to restrict to those integer vectors $\vec{m}$ in each skew-block a for which the parity of m^{2a-1} is equal to that of $n^{2a-1}-n^{2a}+\frac{1+\epsilon_a}{2}$ and the parity of m^{2a} is $\frac{1+\epsilon_a}{2}$, and that terms involving factorials of negative integers are to be omitted from the sum. Note in particular that this basis change shows explicitly how the canonical $\mathbb{Z}_2$-grading of the occupation number basis is induced by the even/odd integer grading of the basis $T_{\vec{n}}$.

As mentioned at the beginning of this section, the two bases $T_{\vec{n}}$ and $\Sigma_{\vec{n}}^{(\epsilon_1\cdots\epsilon_{d/2})}$ are well suited for the representations of the algebra $\mathcal{A}_\theta$ on the Hilbert spaces $\mathcal{H}_{\rm M}$ and $\mathcal{H}_{\rm Q}$, respectively. We will now describe the relationship between these two spaces as well. For this, we shall exploit the Weyl-Wigner correspondence and identify the algebra of functions on non-commutative $\mathbb{R}^d$ with the algebra $\mathcal{A}_\theta(\mathcal{H})$ of Weyl operators on a certain Hilbert space $\mathcal{H}$. The group ${\rm Inn}(\mathcal{A}_\theta)$ of inner automorphisms (gauge transformations) of the algebra $\mathcal{A}_\theta$ is then most efficiently computed via its lift to this Hilbert space as [32]

$${\rm Inn}_{\mathcal{H}}(\mathcal{A}_\theta) = \left\{U\in{\rm U}(\mathcal{H})~\big|~UJ=JU\,,\,\imath_U\in{\rm Inn}\left(\mathcal{A}_\theta(\mathcal{H})\right)\right\}, \tag{3.38}$$

where ${\rm U}(\mathcal{H})$ is the group of unitary endomorphisms of $\mathcal{H}$,

$$\imath_U(F) = U\,F\,U^{-1}\,, \qquad F\in\mathcal{A}_\theta(\mathcal{H})\,, \tag{3.39}$$

and J is the Tomita involution which is defined as the anti-linear, self-adjoint unitary isometry of the Hilbert space $\mathcal{H}$ such that $J\mathcal{A}_\theta(\mathcal{H})J^{-1}=\mathcal{A}'_\theta(\mathcal{H})$ is the commutant of the algebra $\mathcal{A}_\theta$ in $\mathcal{H}$. If $\mathcal{A}_\theta$ acts on $\mathcal{H}$ from the left (resp. right), then $J\,\mathcal{H}$ is a right (resp. left) $\mathcal{A}_\theta$-module. The projection $\pi:{\rm Inn}_{\mathcal{H}}(\mathcal{A}_\theta)\to{\rm Inn}(\mathcal{A}_\theta)$ is given in terms of the Wigner transform as

$$\pi(U) = \Omega^{-1}\,\imath_U\,. \tag{3.40}$$

In the case that $\mathcal{H}=\mathcal{H}_{\rm M}$ is the L^2-completion (2.15) of the algebra $\mathcal{A}_0$, the canonical involution J is just (complex) conjugation,

$$J(\psi) = \psi^\dagger\,, \qquad \forall\psi\in\mathcal{H}_{\rm M}\,. \tag{3.41}$$

In this sense J may be thought of as a charge conjugation operator. When $\mathcal{H} = \mathcal{H}_{\rm Q}$ is the Hilbert space (2.22), i.e. the space of functions of the complex variables z_a, the algebra $\mathcal{A}_\theta(\mathcal{H}_{\rm Q})$ consists of functionals of the operators z_a and $\partial/\partial z_a$, with the representation $z_a \mapsto z_a$ and $z_a^* \mapsto \partial/\partial z_a$ on $\mathcal{H}_{\rm Q}$. The symmetry J then effectively enlarges the Hilbert space $\mathcal{H}_{\rm Q}$ to $\mathcal{H}_{\rm M}$, with $\mathcal{A}'_\theta(\mathcal{H}_{\rm Q})$ the space of functionals of the operators z_a^* and $\partial/\partial z_a^*$. This algebra is naturally isomorphic to $\mathcal{A}_\theta(\mathcal{H}_{\rm Q})$.

The bi-module structure induced by the operator J in (3.38) can be motivated physically within the context of open string quantization in background Neveu-Schwarz fields. Quantizing the point particle at a given endpoint of an open string produces a Hilbert space $\mathcal{H}$. In the Seiberg-Witten scaling limit $\alpha' \, \theta_i{}^j \to \infty$ [3], whereby the string oscillations can be neglected, the imposition of identical boundary conditions at both endpoints of an open string yields a Hilbert space of the form $\mathcal{H} \otimes \mathcal{H}^\vee$, where $\mathcal{H}^\vee$ is the complex conjugate $\mathcal{A}_\theta$-module to $\mathcal{H}$ corresponding to the opposite orientations of a pair of string endpoints. We may naturally identify the Hilbert spaces $\mathcal{H} = \mathcal{H}_{\rm Q}$ [3] and $\mathcal{H}^\vee = J\,\mathcal{H}$. The property in (3.38) then reflects the fact that a lifted gauge transformation should preserve the actions of $\mathcal{A}_\theta$ at opposite ends of the open strings. We can therefore interpret the density matrix representation of the gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$ as that which pertains to a single endpoint of an open string, i.e. the unoriented case of type-I superstrings, while the $\star$-monomial representation is related to type-II superstrings and the presence of both endpoints corresponding to a chosen relative orientation. In this way the quotient by the real structure J maps type-II D-branes onto type-I D-branes and their associated orientifold planes. This has been used in [37] to construct non-commutative gauge algebras based on non-unitary groups.

4. Non-commutative canonical transformations

In this section we will explore the geometric spacetime transformations which are induced by non-commutative gauge transformations. These can all be interpreted in terms of the symplectic geometry induced on $\mathbb{R}^d$ by the constant antisymmetric tensor θ_{ij}. We will see in fact that the gauge algebra of non-commutative Yang-Mills theory is a deformation of the algebra of symplectomorphisms of $\mathbb{R}^d$ in a very precise and analytical way.

4.1 Quantum deformation of the Poisson algebra

As we have discussed in section 2.2, the inner automorphisms of the algebra $\mathcal{A}_\theta$ can be regarded in non-commutative geometry as the counterpart of "point transformations" in a space in which it is not really appropriate to speak of points. Infinitesimal inner automorphisms of the form (2.34) are necessarily volume preserving, since the corresponding spacetime averages transform as

$$\int \mathrm{d}^d x \ f(x) \ \longmapsto \ \int \mathrm{d}^d x \ f(x) + i \int \mathrm{d}^d x \ [\lambda, f]_\star(x) \,. \tag{4.1}$$

Because of (2.5), the integral of the $\star$-commutator in (4.1) vanishes, and so the volume integral of any covariant function is invariant under inner automorphisms. Thus, the volume preserving diffeomorphisms have a very special place in non-commutative gauge theory. They are defined infinitesimally by the transformations $f \mapsto f + \delta_V f$ of functions $f \in \mathcal{A}_0$ by

$$\delta_V f(x) = V(f)(x) = V^i(x)\,\partial_i f(x)\,, \qquad \text{with } \partial_i V^i(x) = 0\,. \tag{4.2}$$

Via an integration by parts we may deduce from the divergence-free condition of (4.2) that $\int \mathrm{d}^d x \ \delta_V f(x) = 0$. Therefore, non-commutative gauge symmetries *cannot* realize arbitrary diffeomorphims, but rather only the subalgebra of volume-preserving transformations of the spacetime. The natural appearance of volume preserving diffeomorphisms, which can be given a brane interpretation [38], is in fact a general feature of the spacetime symmetries induced by non-commutative gauge theory [15].

A generic transformation of the form (4.2) is parametrized by rank $d-2$ tensors $\chi_{i_1 \cdots i_{d-2}}(x)$ as

$$V = \epsilon^{i i_1 \cdots i_{d-1}}\,\partial_{i_{d-1}}\chi_{i_1 \cdots i_{d-2}}(x)\ \partial_i\,. \tag{4.3}$$

The subalgebra $\mathrm{sdiff}(\mathbb{R}^d)$ of symplectic diffeomorphisms comes from taking the vector fields $V = V_F$, where

$$V_F = \theta^{ij}\,\partial_i F\ \partial_j \tag{4.4}$$

for $F \in \mathcal{A}_0$ generates the canonical transformation

$$V_F(f) = \{f, F\}_\theta\,. \tag{4.5}$$

Here

$$\{f, g\}_\theta = \theta^{ij}\,\partial_i f\,\partial_j g \tag{4.6}$$

is the Poisson bracket on the algebra $\mathcal{A}_0$, and the operators (4.4) generate the Poisson-Lie algebra of $\mathbb{R}^d$,

$$[V_F, V_G] = V_{\{F,G\}_\theta}\,. \tag{4.7}$$

We will see in fact that if we identify non-commutative $\mathbb{R}^d$ with the quantum mechanical phase space of a point particle, then the transformations (4.4) induce a Lie algebra of quantum deformed canonical transformations. Similar algebras have been studied within the same context in [39].

In two dimensions the canonical transformations coincide with the area preserving diffeomorphisms of spacetime, but for $d > 2$ they form a proper subalgebra of the algebra of volume preserving diffeomorphisms. Inner automorphisms of the algebra $\mathcal{A}_\theta$ can only generate point transformations which correspond to symplectic diffeomorphisms. Because of the derivation property (2.7), we necessarily have $i\,[x_i,\lambda]_\star = -\theta_i{}^j\,\partial_j\lambda$, and so any gauge function λ generates a canonical transformation of the spacetime coordinates. In fact, it is well known that to first order in θ_{ij} symplectic diffeomorphisms can be realized as inner automorphisms on a Moyal space. This can be seen explicitly by examining the first few terms of the series expansion (3.17) in θ_{ij}, which for $n^i, m^i > 2$ yields

$$
\begin{aligned}
\left[T_{\vec n}\,,\,T_{\vec m}\right]_\star = &\sum_{i\neq j} i\,\theta_{ij}\,n^i\,m^j\;T_{\vec n+\vec m-\vec e_i-\vec e_j} + \\
&+\sum_{i\neq j}\frac{(i\,\theta_{ij})^3}{6}\;\Big[n^j(n^j-1)(n^j-2)m^i(m^i-1)(m^i-2)- \\
&\qquad\qquad -\;n^i(n^i-1)(n^i-2)m^j(m^j-1)(m^j-2)+ \\
&\qquad\qquad +\,3n^in^jm^im^j\left((n^j-1)(m^i-1)-(n^i-1)(m^j-1)\right)\Big]\times \\
&\times T_{\vec n+\vec m-3(\vec e_i+\vec e_j)}+O\left(\theta_{ij}^5\right).
\end{aligned}
\tag{4.8}
$$

It is easy to see that the truncation of the expansion (4.8) to order θ_{ij} satisfies the Jacobi identity. The $\star$-commutation relations to this order close to a Lie algebra which approximates the full gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$. In fact, in each commuting skew-block a we can define the operators

$$
\mathcal{L}^{(a)}_{\vec n} = T^{(a)}_{\vec n+\underline{1}}\,,\qquad n^i\geq -1\,,
\tag{4.9}
$$

and find that the first line of (4.8) realizes the commutation relations of the $W_{1+\infty}$ algebra

$$
\left[\mathcal{L}^{(a)}_{\vec n}\,,\,\mathcal{L}^{(a)}_{\vec m}\right]_\star = 2i\,\left((\vec n+\underline{1}\,)\wedge(\vec m+\underline{1}\,)\right)^{(a)}\;\mathcal{L}^{(a)}_{\vec n+\vec m}+O\left(\theta_a^3\right).
\tag{4.10}
$$

We have defined the antisymmetric bilinear form

$$
\vec n\wedge\vec m = \frac{1}{2}\,n^i\,\theta_{ij}\,m^j
\tag{4.11}
$$

corresponding to the symplectic structure of $\mathbb{R}^d$. Regarding non-commutative $\mathbb{R}^d$ as a symplectic space, the first order $\star$-commutator is in fact the Poisson bracket [40]. This is not true if one truncates the $\star$-commutators at higher orders in θ_{ij}, because while a generic bracket which is a deformation of the Poisson bracket (such as the Moyal bracket) generates a Lie algebra, such deformations are either of order 0 or contain infinitely many terms. Note that the presence of the Tomita involution J effectively doubles the $W_{1+\infty}$ symmetry above to two commuting copies [39], so that the total symmetry algebra is $W_{1+\infty}\otimes W^\vee_{1+\infty}$ with $W^\vee_{1+\infty} = J\,W_{1+\infty}\,J^{-1}$.

Thus, in the limit $\theta_{ij}\to 0$ with the rescaled generators $\theta_i{}^j\,T_{\vec n}$ finite, the non-commutative gauge algebra truncates to the Poisson-Lie algebra of $\mathbb{R}^d$ defined by the Poisson bracket (4.6), i.e. $[T_{\vec n},T_{\vec m}]_\star = i\,\{T_{\vec n},T_{\vec m}\}_\theta$ as $\theta_{ij}\to 0$. In this limit, the

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operators $T_{\vec{n}}$ are the generators

$$W_{\vec{n}} \equiv V_{T_{\vec{n}}} = \sum_{i \neq j} n^i \, \theta_{ij} \, T_{\vec{n}-\vec{e}_i} \; \partial_j \tag{4.12}$$

of symplectic diffeomorphisms which preserve the symplectic geometry of $\mathbb{R}^d$ (equivalently the Poisson bracket (4.6)) and the canonical transformation is an inner automorphism of the algebra $\mathcal{A}_0$. We have found therefore that the gauge algebra, leading to generic volume preserving transformations (4.1), is a *deformation* of the Poisson-Lie algebra on a commutative space. In contrast, for $d > 2$ the Lie algebra of volume preserving diffeomorphisms of $\mathbb{R}^d$ cannot be deformed. The two sorts of gauge transformations agree in the infrared limit of the non-commutative gauge theory, but differ drastically in the ultraviolet regime where the effects of noncommutativity become important. Indeed, because of the UV/IR mixing property of non-commutative field theories [41], the equivalence between the Moyal and Poisson brackets in the commutative limit will cease to be exact at the quantum level. The deformation of diffeomorphisms that occurs here is analogous to the deformations of spacetime symmetries that occur in string theory due to a finite Regge slope α', and indeed the noncommutativity scale plays a role analogous to the string scale [14], being both a regulator and a source of non-locality in the theory. One of the intents of the ensuing analysis is to give a more precise geometric meaning to this deformation, and to describe in exactly what sense it can be understood as a diffeomorphism of flat space $\mathbb{R}^d$. We remark here that the geometrical meaning of the higher order terms in the expansion of (3.17) in θ_{ij} is unclear. The $O(\theta_{ij}^3)$ correction of (4.8) is known to correspond to a symplectic connection, i.e. a connection of a complex line bundle over $\mathbb{R}^d$ which preserves its Poisson structure.

4.2 Algebraic description

In this subsection we will interpret the gauge Lie algebra $\mathrm{u}(\mathcal{A}_\theta)$ of non-commutative electrodynamics as an appropriate completion of the infinite unitary Lie algebra $\mathrm{u}(\infty)$. We have already seen in section 3.3 that gauge transformations can be lifted to inner automorphisms on a given $\mathcal{A}_\theta$-module $\mathcal{H}$. The unitary operators on $\mathcal{H}$ are intimately related to the automorphisms of the non-unital C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators acting on the Hilbert space, i.e. the completion of the algebra of finite-rank operators on $\mathcal{H}$ in the operator-norm

$$\|F\|_\infty = \sup_{(\psi,\psi)_\mathcal{H} \leq 1} \sqrt{(F\psi, F\psi)_\mathcal{H}}\,, \tag{4.13}$$

where in general F is a bounded linear operator on $\mathcal{H}$ and $(\cdot,\cdot)_\mathcal{H}$ denotes the inner product on the Hilbert space. The physical relevance of subalgebras of compact operators lies in the content of the Stone-von Neumann-Mackey theorem [42], which asserts that $\mathcal{H}$ is unitarily equivalent to a direct sum of Schrödinger $\mathcal{A}_\theta$-modules (2.22).

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To determine the gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$, we will thereby focus our attention on the irreducible Schrödinger representation of the algebra $\mathcal{A}_\theta$. The image under the Weyl map Ω of the algebra $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions is contained in the algebra of compact operators on the quantum mechanical Hilbert space $\mathcal{H}_{\rm Q}$.[3] Completing the algebra $\mathcal{A}_\theta$ in the L^∞-norm (2.3) yields an algebra whose Ω-image is the C^*-algebra of compact operators $\mathcal{K}(\mathcal{H}_{\rm Q})$ (see for example [45, Lemma 3.41]). It is important at this point to notice that while the norm completion of the commutative algebra $\mathcal{A}_0$ is the C^*-algebra $C(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d \to \mathbb{C}$, the C^*-completion of $\mathcal{A}_\theta$ is *not* the algebra $C(\mathbb{R}^d)$ equipped with the $\star$-product. In fact, some smoothness restrictions on the functions are required in order that their Weyl maps yield bounded operators [46]. This is certainly true for the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of functions that we are presently working with. The sorts of generalized functions comprising the C^*-completion of the algebra $\mathcal{A}_\theta$ are not known in general.

Nevertheless, for our purposes it will suffice to know that the image of the completion of $\mathcal{A}_\theta$ under the Weyl map Ω is the algebra of compact operators in the case $\mathcal{H} = \mathcal{H}_{\rm Q}$. Furthermore, the Hilbert space $\mathcal{H}_{\rm Q} \otimes \mathcal{H}_{\rm Q}^\vee$, regarded as the space of square-integrable functions of the complex coordinates z and z^*, is precisely the algebra of Hilbert-Schmidt operators on the Hilbert space $\mathcal{H}_{\rm Q}$ [43]. It represents the joining of open string endpoints, and it is dense (in the operator-norm topology) in $\mathcal{K}(\mathcal{H}_{\rm Q})$. Generally, given $U \in \mathrm{U}(\mathcal{H})$, there is the natural automorphism $\imath_U$ of $\mathcal{K}(\mathcal{H})$. The map $U \mapsto \imath_U$ defines a continuous homomorphism (in the operator-norm topology) of $\mathrm{U}(\mathcal{H})$ onto $\mathrm{Aut}(\mathcal{K}(\mathcal{H}))$ with kernel $\mathrm{U}(1)$ corresponding to the phase multiples of the identity operator $\mathbb{1}_{\mathcal{H}}$. This identifies the automorphism group

$$\mathrm{Aut}\left(\mathcal{K}(\mathcal{H})\right) = \mathrm{PU}(\mathcal{H})\,, \tag{4.14}$$

where $\mathrm{PU}(\mathcal{H}) = \mathrm{U}(\mathcal{H})/\,\mathrm{U}(1)$ is the group of projective unitary automorphisms of the Hilbert space $\mathcal{H}$. Note that the defining representation of $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$ is, up to unitary equivalence, the only irreducible representation of the C^*-algebra of compact operators.

For most of the analysis which follows in this subsection it will suffice to consider the Fock space (3.22) of a given skew-block a of the antisymmetric tensor θ_{ij}. At the end of the analysis we can then easily stitch the skew-blocks together to produce the final result. We will thereby effectively work in $d = 2$ dimensions and drop the a indices on all quantities for the most part. We shall therefore start by analysing the algebra $\mathcal{K}(\ell^2(\mathbb{Z}_+))$ of compact operators on the Fock space $\ell^2(\mathbb{Z}_+) = \bigoplus_{n\geq 0} \mathbb{C}|n\rangle$. It is the operator-norm closure

$$\mathcal{K}\left(\ell^2(\mathbb{Z}_+)\right) = \mathcal{C}^*_\infty\left(\mathbb{M}_\infty(\mathbb{C})\right) \tag{4.15}$$

[3] Generally, from (3.9) it follows that if $f \in L^1(\mathbb{R}^d, \mathrm{d}^dx)$, then $\Omega(f)$ is a trace-class operator on $\mathcal{H}$ and hence is compact. If $f \in L^2(\mathbb{R}^d, \mathrm{d}^dx)$ and $\mathcal{H}$ is the Hilbert space (2.22) of quantum mechanics, then $\Omega(f)$ is a Hilbert-Schmidt operator on $\mathcal{H}_{\rm Q}$ [43]. For further discussion of larger classes of functions which have compact Weyl transforms, see [44].

of the algebra $\mathbb{M}_\infty(\mathbb{C})$ of finite-rank operators acting on $\ell^2(\mathbb{Z}_+)$. The latter algebra is the inductive $N \to \infty$ limit

$$\mathbb{M}_\infty(\mathbb{C}) = \bigcup_{N=1}^{\infty} \mathbb{M}_N(\mathbb{C}) \tag{4.16}$$

of finite dimensional, $N \times N$ matrix algebras $\mathbb{M}_N(\mathbb{C}) = \mathcal{K}(\mathbb{C}^N)$. As a set, (4.16) is made of coherent sequences with respect to the natural system of embeddings

$$\begin{aligned} \mathbb{M}_N(\mathbb{C}) \;&\hookrightarrow\; \mathbb{M}_{N+1}(\mathbb{C})\,, \\ M \;&\longmapsto\; \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.17}$$

The unitary group of the algebra $\mathbb{M}_N(\mathbb{C})$ is just the usual $N \times N$ unitary group $\mathrm{U}(N)$, and the map $\imath : \mathrm{U}(N) \to \mathrm{Inn}(\mathbb{M}_N(\mathbb{C}))$ has kernel $\ker \imath = \mathrm{U}(1)$. The group of finite-dimensional inner automorphisms is thereby given as

$$\mathrm{Inn}\left(\mathbb{M}_N(\mathbb{C})\right) = \frac{\mathrm{U}(N)}{\mathrm{U}(1)} = \frac{\mathrm{SU}(N)}{\mathbb{Z}_N}\,. \tag{4.18}$$

The large-N completion of (4.18) then coincides with the automorphism group (4.14) for the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}_+)$. In the following we will not deal with any global aspects of the gauge groups and focus only on the infinitesimal gauge transformations. From the preceding arguments it is then clear that the gauge algebra of (4.15) is the operator-norm closure

$$u\left[\mathcal{K}\left(\ell^2(\mathbb{Z}_+)\right)\right] = \mathcal{C}^*_\infty\left(\mathrm{u}(\infty)\right), \tag{4.19}$$

where

$$\mathrm{u}(\infty) = \bigcup_{N=1}^{\infty} \mathrm{u}(N) \tag{4.20}$$

is the infinite-dimensional Lie algebra of finite-rank hermitean operators on Fock space.

The result (4.19) illustrates how the U(1) non-commutative gauge theory contains all non-abelian unitary gauge groups. In the large-N limit of (4.18) leading to (4.14), we replace the global center subgroup $\mathbb{Z}_N$ of $\mathrm{SU}(N)$ by the phase group U(1). The *local* gauge group, or equivalently the gauge algebra, is thereby intimately related to the infinite-dimensional Lie algebra $\mathrm{su}(\infty)$ of traceless finite-rank hermitean operators. In fact, the description of this subsection can be related to the standard Cartan basis of $\mathrm{su}(\infty)$ [47] by using the presentation of the gauge algebra of section 3.2. For this, we introduce the step operators

$$\hat{E}_{\vec{n}} = |n^1\rangle\langle n^2| = \frac{1}{2}\left(\hat{\Sigma}^-_{\vec{n}} + i\,\hat{\Sigma}^+_{\vec{n}}\right) \tag{4.21}$$

for each two-vector $\vec{n} = (n^1, n^2)$ of non-negative integers. The operators (4.21) are orthonormal in the standard inner product on the space $\mathbb{M}_\infty(\mathbb{C})$,

$$\mathrm{Tr}_\infty \left(\hat{E}_{\vec{n}}^\dagger \, \hat{E}_{\vec{m}} \right) = \delta_{\vec{n}\,\vec{m}} \ , \tag{4.22}$$

and they thereby span the linear space of finite-rank operators on $\ell^2(\mathbb{Z}_+)$. The Cartan subalgebra of su(∞) is then taken to be generated by the traceless diagonal operators

$$\hat{H}_n = \hat{E}_{\underline{n}} - \hat{E}_{\underline{n+1}} \, , \qquad n \in \mathbb{Z}_+ \, . \tag{4.23}$$

The simple root vectors $\vec{\alpha}_n$ have components defined by

$$\left[\hat{H}_m \, , \, \hat{E}_{\underline{n}+\vec{e}_2} \right] = \left(\vec{\alpha}_n\right)_m \, \hat{E}_{\underline{n}+\vec{e}_2} \ , \tag{4.24}$$

so that

$$\left(\vec{\alpha}_n\right)_m = -\delta_{n-1,m} + 2\,\delta_{nm} - \delta_{n+1,m} \, . \tag{4.25}$$

The Fock space representation of the infinite-dimensional Lie algebra su(∞) may thereby be given as the semi-infinite Dynkin diagram

$$\bigcirc\!\!-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\!\!\bigcirc\!\!-\!\!\!- \ \cdots\cdot \tag{4.26}$$

The Lie algebra u($\mathcal{A}_\theta$) is nothing but a "Schwartzian" version of su(∞), with which it shares the same basis.

We must remember however that we are dealing here with infinite-dimensional algebras, and that therefore the identifications have to be made with great care. While generally for su(∞) what is meant is the space of arbitrarily large but *finite* matrices, we are using Schwartzian but otherwise infinite combinations of the basis elements (4.21). If one restricts the Lie algebras to *finite* combinations of the generators, then the two presentations that we have constructed in section 3 would not be representations of the same algebra because the change of basis described in section 3.3 involves infinite (Schwartzian) series. Moreover, there are infinitely many (non-mutually pairwise isomorphic) versions of su(∞) which depend on the way that the large-N limit is taken [48]. Indeed, there are many distinct algebras that can be obtained from $\mathbb{M}_N(\mathbb{C})$ by taking more complicated embeddings than the simplest, canonical one (4.17) (see [49] for the example of the non-commutative torus). The connections between su(∞) and the area preserving diffeomorphisms of a two-dimensional torus are of a similar kind. In that case the two Lie algebras are not isomorphic [50]. However, their difference lies in the different limits, or equivalently in the high frequency components, and for a variety of physical applications the difference will not be relevant from the point of view of infinitesimal gauge transformations. This discussion clarifies the explicit connection between non-commutative U(1) gauge theories and su(∞), and hence matrix models.

4.3 Geometrical description

We will now pass from an algebraic description of the gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$ to a more geometrical one. For this, we note that the Weyl operators (3.6) should really be regarded as elements of the Heisenberg group algebra, which is locally generated by the unitary translation operators

$$\hat{\mathcal{T}}(\xi) = \exp\left(i\,\xi\cdot\hat{x}\right) = \sum_{\vec{n}\in\mathbb{Z}_+^d} \frac{(-1)^{|\vec{n}|/2}\,(\xi^1)^{n^1}\cdots(\xi^d)^{n^d}}{n^1!\cdots n^d!}\,\hat{T}_{\vec{n}}\,, \tag{4.27}$$

where $\xi^i\in\mathbb{R}$. They enjoy the properties

$$\hat{\mathcal{T}}(\xi)^\dagger = \hat{\mathcal{T}}(-\xi)\,, \tag{4.28}$$

$$\hat{\mathcal{T}}(\xi)\,F(\hat{x})\,\hat{\mathcal{T}}(\xi)^\dagger = F(\hat{x}+\theta\,\xi)\,, \tag{4.29}$$

$$\hat{\mathcal{T}}(\xi)\,\hat{\mathcal{T}}(\xi') = \mathrm{e}^{-i\,\xi\wedge\xi'}\,\hat{\mathcal{T}}(\xi+\xi')\,, \tag{4.30}$$

where $F\in\mathcal{A}_\theta(\mathcal{H})$. In the skew-diagonalization we may use (3.21) to equivalently write

$$\hat{\mathcal{T}}(\xi) = \exp\left(i\,\xi_z^a\,\hat{z}_a + i\,\xi_z^{a\,*}\,\hat{z}_a^\dagger\right) \tag{4.31}$$

as operators on the quantum mechanical Hilbert space $\mathcal{H}_{\rm Q}$, where

$$\xi_z^a = \sqrt{\frac{\theta_a}{2}}\,\left(\xi^{2a-1} - i\,\xi^{2a}\right)\,. \tag{4.32}$$

We may thereby associate to the translation operators the coherent states

$$|\xi_z\rangle = \hat{\mathcal{T}}(\xi)|\vec{0}\,\rangle\,, \tag{4.33}$$

where $|\vec{0}\,\rangle = |0\rangle\otimes\cdots\otimes|0\rangle$ is the $\frac{d}{2}$-fold Fock vacuum state of the Hilbert space (2.22). We recall from section 2.2 that the subspace spanned by the coherent states (4.33) is precisely one of the irreducible components of the $\mathcal{A}_\theta$-module $\mathcal{H}_{\rm M}$ of fundamental matter fields.

From (4.30) we may infer the commutation relations

$$\left[\hat{\mathcal{T}}(\xi)\,,\,\hat{\mathcal{T}}(\xi')\right] = -2i\,\sin\left(\xi\wedge\xi'\right)\;\hat{\mathcal{T}}(\xi+\xi')\,. \tag{4.34}$$

The advantage of passing to the Heisenberg-Weyl group is that it admits (in the compact case) finite-dimensional representations which can be used to map the unitary group representations above to geometric ones. Namely, when $\theta=M/N$, with $M,N\in\mathbb{Z}_+$ coprime, and the Fourier momenta ξ are restricted to lie on the lattice $\mathbb{Z}_+^d$, then there is an N-dimensional unitary representation of the commutation relations (4.34). This restriction on θ, once a large-N limit is taken, does not affect expectation values of operators nor any physically measurable quantities [49].

The restriction on the momenta means that the space we are effectively considering is a (non commutative) torus. Since we are concerned here only with infinitesimal gauge transformations, one might think that the change from the compact to the non-compact case will not alter the structure of the Lie algebra $\mathrm{u}(\mathcal{A}_\theta)$. This is not precisely correct, because in non-commutative quantum field theory there is an interplay between local (ultraviolet) and global (infrared) characteristics. We shall see in fact that the gauge Lie algebra we find in this case is not exactly the one that we found in the previous subsection.

For simplicity we shall assume that N is odd. For each $\vec{n} = (n^1, n^2) \in \mathbb{Z}_+^2$, we introduce the $N \times N$ unitary unimodular matrices

$$\hat{\mathcal{W}}_{\vec{n}}^{(N)} = \omega^{n^1 n^2/2} \left(\hat{\Gamma}_{\rm c}\right)^{n^1} \left(\hat{\Gamma}_{\rm s}\right)^{n^2} , \tag{4.35}$$

where $\omega = {\rm e}^{4\pi i\, M/N}$ is an $N^{\rm th}$ root of unity. The matrices $\hat{\Gamma}_{\rm c}$ and $\hat{\Gamma}_{\rm s}$ generate the Weyl-'t Hooft algebra

$$\hat{\Gamma}_{\rm c}\, \hat{\Gamma}_{\rm s} = \omega\, \hat{\Gamma}_{\rm s}\, \hat{\Gamma}_{\rm c} \,, \tag{4.36}$$

which, up to a gauge transformation $\hat{\Gamma}_i \mapsto U\, \hat{\Gamma}_i\, U^{-1}$ with $U \in \mathrm{SU}(N)$, may be uniquely represented by the usual clock and cyclic shift operators for $\mathrm{SU}(N)$,

$$\begin{aligned} \hat{\Gamma}_{\rm c} &= \sum_{n=0}^{N-1} \omega^n\, |n\rangle\langle n| = -\sum_{n\geq 0} \omega^n \sum_{m=0}^{n-1} \hat{H}_m \,, \\ \hat{\Gamma}_{\rm s} &= \sum_{n=0}^{N-1} |n+1\,(\mathrm{mod}\, N)\rangle\langle n| = \sum_{n\geq 0} \hat{E}_{\underline{n}+\vec{e}_1\,(\mathrm{mod}\ N)} \,. \end{aligned} \tag{4.37}$$

The finite-rank operators (4.35) satisfy the unitarity condition

$$\hat{\mathcal{W}}_{\vec{n}}^{(N)\,\dagger} = \hat{\mathcal{W}}_{-\vec{n}}^{(N)} \,, \tag{4.38}$$

and, because $(\hat{\Gamma}_i)^N = \mathbb{1}_N$, they are periodic in n^i modulo N,

$$\hat{\mathcal{W}}_{\vec{n}+N\vec{e}_i}^{(N)} = \hat{\mathcal{W}}_{\vec{n}}^{(N)} \,, \qquad i = 1, 2 \,. \tag{4.39}$$

The property (4.39) implies that there are only N^2 distinct operators (4.35) corresponding to the values $\vec{n} \in \mathbb{Z}_N^2$. Furthermore, they are orthonormal in the standard inner product on the space of complex matrices,

$$\mathrm{Tr}_N \left(\hat{\mathcal{W}}_{\vec{n}}^{(N)\,\dagger}\, \hat{\mathcal{W}}_{\vec{m}}^{(N)} \right) = \delta_{\vec{n}\,\vec{m}} \,. \tag{4.40}$$

It follows that the underlying linear space of the C^*-algebra $\mathbb{M}_N(\mathbb{C})$ is spanned by the matrices (4.35). This is known as the Weyl basis for the Lie algebra $gl(N, \mathbb{C})$.

In fact, from (4.40) it follows that the N^2-1 matrices $\hat{\mathcal{W}}^{(N)}_{\vec{n}}$, $\vec{n}\neq\vec{0}$, form a complete set of traceless unitary matrices. By taking real and imaginary combinations, they therefore span the Lie algebra su(N). From (4.36) we also find that these operators close projectively under multiplication,

$$\hat{\mathcal{W}}^{(N)}_{\vec{n}}\,\hat{\mathcal{W}}^{(N)}_{\vec{m}}=\omega^{\vec{m}\wedge\vec{n}}\,\hat{\mathcal{W}}^{(N)}_{\vec{n}+\vec{m}\,(\mathrm{mod}\ \underline{N})}\,, \tag{4.41}$$

which specifies an additive grading structure. The relation (4.41) further leads to an explicit form for the su(N) structure constants in the Weyl basis through the commutation relations of the Fairlie-Fletcher-Zachos trigonometric algebra [13]

$$\left[\hat{\mathcal{W}}^{(N)}_{\vec{n}}\,,\,\hat{\mathcal{W}}^{(N)}_{\vec{m}}\right]=\frac{i\,N}{2\pi M}\,\sin\left(\frac{4\pi M}{N}\,\vec{n}\wedge\vec{m}\right)\,\hat{\mathcal{W}}^{(N)}_{\vec{n}+\vec{m}\,(\mathrm{mod}\ \underline{N})}\,, \tag{4.42}$$

where we have rescaled the operators $\hat{\mathcal{W}}^{(N)}_{\vec{n}}\mapsto\frac{N}{4\pi M}\,\hat{\mathcal{W}}^{(N)}_{\vec{n}}$. The explicit transformation between the Weyl and Cartan bases of su(N) can be obtained from (4.22), (4.37) and (4.40) to give

$$\begin{aligned}\hat{\mathcal{W}}^{(N)}_{\vec{n}}&=-\frac{N}{4\pi M}\,\omega^{n^2(n^1+n^2+1)/2}\sum_{m=0}^{N-1}\omega^{m(n^1+n^2)}\,\hat{E}_{\underline{m}-n^2\vec{e}_2\,(\mathrm{mod}\ N)}\,,\\ \hat{E}_{\vec{n}\,(\mathrm{mod}\ \underline{N})}&=-\frac{4\pi M}{N}\,\omega^{(n^1-n^2)(n^1-2n^2-1)/2}\sum_{m=0}^{N-1}\omega^{m(n^1-2n^2)/2}\,\hat{\mathcal{W}}^{(N)}_{m\vec{e}_1+(n^1-n^2)\vec{e}_2}\,.\end{aligned} \tag{4.43}$$

We see therefore that the finite-dimensional operators (4.35) formally possess the same algebraic properties as the translation generators (4.27), and can thereby be thought of as a certain $N\times N$ approximation to them. Passing to the inductive large-N limit yields generators $\hat{\mathcal{W}}^{(\infty)}_{\vec{n}}$, $\vec{n}\neq\vec{0}$, of su(∞) which from (4.42) satisfy the commutation relations for the classical W_∞ algebra

$$\left[\hat{\mathcal{W}}^{(\infty)}_{\vec{n}}\,,\,\hat{\mathcal{W}}^{(\infty)}_{\vec{m}}\right]=2i\,\vec{n}\wedge\vec{m}\,\hat{\mathcal{W}}^{(\infty)}_{\vec{n}+\vec{m}}\,. \tag{4.44}$$

Again there are actually two commuting copies of this W_∞ algebra, so that the total symmetry algebra is $W_\infty\otimes W_\infty^\vee$. The W_∞ component acts separately on each Landau level of the oscillator space (3.22), while $W_\infty^\vee$ mixes the different levels but acts in a simple way on the coherent states (4.33).

The algebra (4.44) can be identified with the Lie algebra of the vector fields

$$\hat{\mathcal{W}}^{(\infty)}_{\vec{n}}=i\,V_{\mathcal{T}_{\vec{n}}}=\mathcal{T}_{\vec{n}}\ \vec{n}\wedge\partial\,, \tag{4.45}$$

where $\mathcal{T}_{\vec{n}}(x)=\mathrm{e}^{\,i\,\vec{n}\cdot x}$ are the complete set of harmonics on a two-dimensional square torus. However, from the point of view of infinitesimal diffeomorphisms, this torus basis is simply a matter of convenience. Locally, the canonical transformations generated by (4.45) are simply those which preserve the symplectic two-form $\mathrm{d}x\wedge\mathrm{d}x$, and this property holds whether we are speaking of the torus or all of $\mathbb{R}^d$. This is evident in particular from the natural local isomorphism between the $W_{1+\infty}$ algebra (4.10) and the W_∞ algebra (4.44). The global group structure is a somewhat more subtle issue which will not be dealt with in this paper. From (4.43) we find the explicit relationship between the symplectomorphism generators (4.45) in the torus basis and the Cartan basis of $\mathrm{su}(\infty)$ given in the previous subsection. Again, it should be stressed that the large-N limit of (4.42) leading to (4.44) implies ignoring periodicity and the boundaries of the Brillouin zone in momentum space. This has no bearing however on the local aspects of the problem.

By including the local translation generators $\hat{\mathcal{W}}_{\vec{0}}^{(\infty)}$ which are excluded from (4.45) (the torus has only global translational isometries), we find an extra U(1) symmetry on the plane and hence the Lie algebra $\mathrm{u}(\infty)$. At the infinitesimal level this extra $\mathrm{u}(1)$ factor, which emerges from the infrared properties of the torus, is somewhat "sterile" because it corresponds to a mode which $\star$-commutes with all fields. It can however have consequences for the global properties of the theory. We thereby find an explicit relationship between the infinite unitary Lie algebra $\mathrm{u}(\infty)$ and the symplectomorphism algebra, and hence with the gauge algebra (4.19). Let us emphasize again that the explicit connection made in this section between the non-commutative gauge algebra and the Lie algebra (4.20) holds strictly only in the truncation to finite rank operators. In particular, because of the boundary effects described above, there is no immediate isomorphism with the group of symplectomorphisms of $\mathbb{R}^d$. Nevertheless, all of these algebras complete in norm to the same subalgebra of compact operators if the maps are chosen appropriately.

These properties evidently all persist when gluing the skew-blocks together again to produce the full Hilbert space (2.22), giving the unitary gauge algebra of the entire non-commutative space, $\mathrm{u}(\mathcal{A}_\theta) = \mathcal{C}^*_\infty(\mathrm{u}(\infty)\otimes\mathbb{R}^{d/2}) \cong \mathcal{C}^*_\infty(\mathrm{u}(\infty))$. The main conclusion of this section is then that the gauge algebra of Yang-Mills theory on a Moyal space is a deformation of the Poisson-Lie algebra of $\mathbb{R}^d$ which, for the dense class of functions we are considering, is locally isomorphic to $\mathrm{u}(\infty)$ in the norm closure,

$$\mathrm{u}(\mathcal{A}_\theta) = \mathcal{C}^*_\theta\left(\mathrm{sdiff}(\mathbb{R}^d)\right) \cong \mathcal{C}^*_\infty\left(\mathrm{u}(\infty)\right). \tag{4.46}$$

Here $\mathcal{C}^*_\theta$ denotes the C^*-completion for the Moyal bracket, which defines a subspace of generalized functions on $\mathbb{R}^d$ whose Moyal brackets induce the appropriate deformation of the symplectomorphism algebra $\mathrm{sdiff}(\mathbb{R}^d)$. The identification (4.46) is very natural in light of the intimate relationship that exists between large-N matrix models and non-commutative gauge theory via the Eguchi-Kawai reduction [6, 7, 11]. Indeed, reduced models transmute spacetime degrees of freedom into internal matrix ones. It is for this reason, for example, that M2-brane dynamics arise in Matrix theory [51], whereby the symplectic symmetry appears as a discretization of the residual gauge symmetry of the 11-dimensional supermembrane [52]. This symmetry is also related to the equivalence between large-N reduced models and the Schild model [53]. The generic connection with volume preserving diffeomorphisms is at the very heart of the way that non-commutative gauge theory effectively encodes the target space symmetries of D-branes [15]. The identification (4.46) has also been noted directly in the context of D-brane field theory in [54, 55].

5. Spacetime symmetry algebras

One of the most interesting features of gauge theory on a non-commutative space is the interplay between gauge and spacetime transformations. It is therefore interesting to study subalgebras of the gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$ and to see what sorts of geometric interpretations they admit. There can in fact be situations in which the non-commutative gauge symmetry is broken down to a subalgebra of $\mathrm{u}(\mathcal{A}_\theta)$, due to an action functional which has a minimum that is invariant only under the subalgebra (or due to some other symmetry breaking mechanism). In this case only the gauge transformations of this subalgebra are available, and in effect reduce the spacetime. We shall return to this point in the next section.

In the previous section we saw that local gauge transformations are intimately related to automorphisms of the algebra $\mathcal{K}(\mathcal{H})$ of compact operators (with respect to an appropriate Weyl representation). From this feature we identified $\mathrm{u}(\mathcal{A}_\theta)$ as the closure of the infinite unitary Lie algebra $\mathrm{u}(\infty)$ which contains, in particular, as subalgebras all finite-dimensional unitary gauge algebras $\mathrm{u}(N)$. It contains them in fact several times in various different guises. There are also numerous infinite-dimensional subalgebras of $\mathrm{u}(\mathcal{A}_\theta)$. In this section we shall present a rather elementary description of some of these subalgebras from the point of view of their actions on spacetime scalar fields. In the next section we will describe their interpretations as transformations of the gauge fields. Often we will describe only the two-dimensional case corresponding to a given skew-block of the deformation matrix θ_{ij}, the stitching together of the blocks in the end being straightforward.

The basic idea behind the emergence of Lie subalgebras of $\mathrm{u}(\mathcal{A}_\theta)$ is as follows. Let $\mathcal{G}$ be an abstract n-dimensional Lie algebra with structure constants ${f_{ab}}^c$ in a chosen basis. Let $\mathcal{G}^*$ be its dual, with corresponding generators $X_1,\dots,X_n$ and Lie bracket $[\,\cdot\,,\,\cdot\,]_{\mathcal{G}^*}$,

$$[X_a,X_b]_{\mathcal{G}^*}={f_{ab}}^c\,X_c\,. \tag{5.1}$$

The Lie algebra $\mathcal{G}$ may be realized not only in terms of operators, but also in terms of functions on Poisson manifolds with the Lie bracket replaced by the Poisson bracket [56]. We seek a realization of $\mathcal{G}$ as an n-dimensional Lie subalgebra $\mathcal{G}_\theta$ of the non-commutative gauge algebra $\mathrm{u}(\mathcal{A}_\theta)$. For this, we will construct a map $\rho:\mathbb{R}^d\to\mathcal{G}^*$ whose pullbacks generate the Moyal brackets

$$\left[\rho^*(X_a)\,,\,\rho^*(X_b)\right]_\star=i\,{f_{ab}}^c\,\rho^*(X_c)+O(\theta_{ij})\,. \tag{5.2}$$

The map ρ need not be a projection, and indeed in many of the cases that we shall consider we can take $d < \dim \mathcal{G}^*$. The $O(\theta_{ij})$ terms in (5.2) will generically arise because, as discussed extensively in the previous section, the Moyal bracket yields a deformation of the Poisson bracket of $\mathbb{R}^d$. We will see that the ensuing gauge transformations (inner automorphisms) yield the anticipated diffeomorphisms of spacetime in these cases. However, because the gauge algebra (4.46) actually contains the Lie algebra $\mathrm{sdiff}(\mathbb{R}^d)$ of canonical transformations, we would expect to be able to find maps ρ which lead to undeformed representations of the Lie algebra $\mathcal{G}$. We shall find that this is indeed the case. But because the Moyal bracket modifies non-trivially the C^*-completion in (4.46), the corresponding inner automorphisms will not generate ordinary diffeomorphisms, nor will they admit a natural geometrical interpretation in terms of non-commutative gauge transformations. The following analysis will therefore make the meaning of the deformation encountered in the previous section somewhat more precise, at least from a geometrical point of view.

5.1 Gaussian algebras

Let us begin with the simplest illustrative example, which involves the "momentum" operators

$$P_i = -\sum_{j=1}^{d} \theta_i^{\ j}\, T_{\vec{e}_j} \tag{5.3}$$

that act on the algebra $\mathcal{A}_\theta$ by the inner automorphisms[4]

$$i\,[P_i, f]_\star = \partial_i f\,, \qquad \forall f \in \mathcal{A}_\theta\,, \tag{5.4}$$

and thereby correspond to spacetime translation operators. They generate the non-commutative translation algebra

$$[P_i, P_j]_\star = i\, B_{ij}\,, \tag{5.5}$$

where

$$B_{ij} = \delta_{ik}\, \theta_j^{\ k} \tag{5.6}$$

is a constant background magnetic field. This central extension of the usual abelian Lie algebra of translations of $\mathbb{R}^d$ can be understood in terms of the corresponding group elements (4.27). From (4.30) it follows that they only generate a projective representation of the abelian group of translations in spacetime. We can view this as a genuine representation of a larger group by including the projective phase factors, or equivalently by augmenting the non-abelian Lie algebra (5.5) by the central elements

[4] As discussed in section 2, elements of the form (5.3) morally only generate inner automorphisms of the ring $\mathbb{C}[[x_1,\dots,x_d]]/\mathcal{R}_\theta$. In the following we will not be pedantic about the completions to spaces of Schwartz functions and still refer to transformations of the form (5.4) as inner automorphisms of the algebra $\mathcal{A}_\theta$.

appearing on the right-hand side. We recall from section 3.1 that the $T_{\vec{n}}$'s constitute a basis for $\mathrm{u}(\mathcal{A}_\theta)$ in the enveloping algebra of this Lie algebra. As we will see in the next section, such central extensions have no consequences for the gauge-invariant dynamics of the field theory.

The translation group can be completed into the euclidean group ISO(d) in d dimensions. For illustration, we will first demonstrate this for the case of the Moyal plane $\mathbb{R}^2$ with noncommutativity parameter θ. By defining the operator

$$L_{12} = \frac{1}{2\theta} \left(T_{2\vec{e}_1} + T_{2\vec{e}_2}\right) , \tag{5.7}$$

we see from (3.17) that along with the $\star$-commutation relations (5.5) we obtain a deformed version of the euclidean group ISO(2),

$$[L_{12}, P_1]_\star = i\, P_2 \,, \qquad [L_{12}, P_2]_\star = -i\, P_1 \,. \tag{5.8}$$

That L_{12} can be identified as a rotation generator may be seen as follows. For an arbitrary function

$$f = \sum_{\vec{n}\in\mathbb{Z}^2_+} f_{\vec{n}}\, T_{\vec{n}} \tag{5.9}$$

of the non-commutative algebra $\mathcal{A}_\theta$, the corresponding inner automorphism is given from (3.17) by

$$i\,[L_{12}, f]_\star = \sum_{\vec{n}\in\mathbb{Z}^2_+} f_{\vec{n}} \, \left(n^2\, T_{\vec{n}+\vec{e}_1-\vec{e}_2} - n^1\, T_{\vec{n}+\vec{e}_2-\vec{e}_1}\right) , \tag{5.10}$$

which is readily seen to be the anticipated form of an infinitesimal rotation of a scalar field in two dimensions.

The generalization to arbitrary spacetime dimension d is straightforward. We introduce the $d(d-1)/2$ angular momentum operators

$$L_{ij} = \frac{1}{2} \sum_{k=1}^{d} \left(\theta_i{}^k\, T_{\vec{e}_j+\vec{e}_k} - \theta_j{}^k\, T_{\vec{e}_i+\vec{e}_k}\right) , \tag{5.11}$$

and find, by using (3.17), that along with (5.5) they satisfy the commutation relations of a deformed, non-commutative iso(d) Lie algebra,

$$\begin{aligned}
[L_{ij}, P_k]_\star &= \frac{i}{2}\,(\delta_{ik}\, P_j - \delta_{jk}\, P_i) + \frac{i}{2}\,\left(\theta_{ik}\, \theta_j{}^l - \theta_{jk}\, \theta_i{}^l\right)\, P_l \,, \\
[L_{ij}, L_{kl}]_\star &= \frac{i}{2}\,(\delta_{il}\, L_{jk} + \delta_{jk}\, L_{il} - \delta_{ik}\, L_{jl} - \delta_{jl}\, L_{ik}) + \\
&\quad + \frac{i}{2}\,\Big[\left(\theta_{jl}\, \theta_k{}^m - \theta_{jk}\, \theta_l{}^m\right)\, L_{im} + \left(\theta_{ik}\, \theta_l{}^m - \theta_{il}\, \theta_k{}^m\right)\, L_{jm} + \\
&\qquad + \left(B_{il}\, \theta_j{}^m - B_{jl}\, \theta_i{}^m\right)\, L_{km} + \left(B_{jk}\, \theta_i{}^m - B_{ik}\, \theta_j{}^m\right)\, L_{lm} +
\end{aligned}$$

$$
\begin{aligned}
&+\left(\delta_{il}\,\theta_j{}^m\,\theta_k{}^n+\delta_{jk}\,\theta_i{}^m\,\theta_l{}^n-\delta_{ik}\,\theta_j{}^m\,\theta_l{}^n-\delta_{jl}\,\theta_i{}^m\,\theta_k{}^n\right)\,L_{mn}\Big]+\\
+\frac{i}{4}\,\Bigg[&\theta_{ik}\,T_{\vec e_j+\vec e_k}+\theta_{jl}\,T_{\vec e_i+\vec e_k}-\theta_{il}\,T_{\vec e_j+\vec e_k}-\theta_{jk}\,T_{\vec e_i+\vec e_l}+\\
&+\sum_{m=1}^{d}\Big\{\left(\delta_{il}\,\theta_j{}^m-\delta_{jl}\,\theta_i{}^m\right)\,T_{\vec e_k+\vec e_m}+\left(\delta_{jk}\,\theta_i{}^m-\delta_{ik}\,\theta_j{}^m\right)\,T_{\vec e_l+\vec e_m}+\\
&+\left(\delta_{jk}\,\theta_l{}^m-\delta_{jl}\,\theta_k{}^m\right)\,T_{\vec e_i+\vec e_m}+\left(\delta_{il}\,\theta_k{}^m-\delta_{ik}\,\theta_l{}^m\right)\,T_{\vec e_i+\vec e_m}+\\
&+\sum_{n=1}^{d}\left(B_{ik}\theta_j^m\theta_l^n+B_{jl}\theta_i^m\theta_k^n-B_{il}\theta_j^m\theta_k^n-B_{jk}\theta_i^m\theta_l^n\right)T_{\vec e_m+\vec e_n}\Big\}\Bigg]\,.
\end{aligned}
\tag{5.12}
$$

In both sets of $\star$-commutation relations of (5.12) the first sets of terms on the right-hand sides represent the standard Lie algebra of the euclidean group in d dimensions, while the second terms represent a quantum deformation of this algebra. However, the $\star$-commutators of angular momentum operators do not close to a Lie algebra. There is again an extension of the usual $\mathrm{iso}(d)$ algebra, this time by operators which are non-central elements. We shall see below how this non-central extension may be understood as a generalization of the projective representation (4.30) to the full euclidean group.

The non-commutative translation algebra can also be completed into the Poincaré algebra $\mathrm{iso}(1,d-1)$. In the two-dimensional case this structure arises from the generator

$$
K_{12}=\frac{1}{\theta}\,T_{\vec e_1+\vec e_2}\,, \tag{5.13}
$$

which from (3.17) yields the $\star$-commutation relations

$$
[K_{12},P_1]_\star=i\,P_1\,,\qquad [K_{12},P_2]_\star=-i\,P_2\,. \tag{5.14}
$$

The fact that the gauge algebra of non-commutative Yang-Mills theory contains deformations of both the euclidean and Poincaré algebras is indicative of the general fact that the signature of a non-commutative space is a delicate issue from the field theoretical point of view. Indeed, quantum field theories with non-commuting time coordinate are not unitary and the corresponding Seiberg-Witten limit in these cases yields a model that only makes sense in string theory [57].

These algebraic results all follow from the fact that these subalgebras are themselves part of the larger subalgebra of $\mathrm{u}(\mathcal{A}_\theta)$ consisting of functions which are at most quadratic in the spacetime coordinates. For these functions, the $\star$-commutator truncates to linear order in the deformation parameters θ_{ij}, i.e. the Moyal and Poisson brackets coincide, and it generates a linear canonical transformation of the coordinates x_i. Since

$$
\deg\,\{f,g\}_\theta=\deg\,f+\deg\,g-2 \tag{5.15}
$$

for any two polynomial functions f and g, it follows that the polynomials of degree 2 form a Lie subalgebra in the Moyal bracket. Explicitly, from (3.17) we arrive, in addition to (5.5), at the $\star$-commutation relations

$$\begin{aligned}
\big[T_{2\vec{e}_i}\,,\,T_{2\vec{e}_j}\big]_\star &= 4i\,\theta_{ij}\,T_{\vec{e}_i+\vec{e}_j}\,, \qquad (\text{no sum on } i,j)\,,\\
\big[T_{2\vec{e}_i}\,,\,T_{\vec{e}_j+\vec{e}_k}\big]_\star &= 2i\,\theta_{ij}\,T_{\vec{e}_i+\vec{e}_k}+2i\,\theta_{ik}\,T_{\vec{e}_i+\vec{e}_j}\,, \qquad (\text{no sum on } i)\,,\\
\big[T_{\vec{e}_i+\vec{e}_j}\,,\,T_{\vec{e}_k+\vec{e}_l}\big]_\star &= i\,\theta_{ik}\,T_{\vec{e}_j+\vec{e}_l}+i\,\theta_{il}\,T_{\vec{e}_j+\vec{e}_k}+i\,\theta_{jk}\,T_{\vec{e}_i+\vec{e}_l}+i\,\theta_{jl}\,T_{\vec{e}_i+\vec{e}_k}\,,\\
\big[T_{\vec{e}_i}\,,\,T_{2\vec{e}_j}\big]_\star &= 2i\,\theta_{ij}\,T_{\vec{e}_j}\,, \qquad (\text{no sum on } j)\,,\\
\big[T_{\vec{e}_i}\,,\,T_{\vec{e}_j+\vec{e}_k}\big]_\star &= i\,\theta_{ij}\,T_{\vec{e}_k}+i\,\theta_{ik}\,T_{\vec{e}_j}\,.
\end{aligned} \tag{5.16}$$

In addition to the deformed euclidean and Poincaré algebras above, there are a multitude of other subalgebras which can be readily deduced from (5.16). For instance, in two spacetime dimensions, it is easily seen from (5.16) that the full set of quadratic operators

$$\begin{aligned}
J_3 &= \frac{1}{\theta}\,T_{\vec{e}_1+\vec{e}_2}\,,\\
J_+ &= \frac{1}{2\theta}\,T_{2\vec{e}_2}\,,\\
J_- &= -\frac{1}{2\theta}\,T_{2\vec{e}_1}
\end{aligned} \tag{5.17}$$

in this case yield a realization of the (undeformed) sl$(2,\mathbb{R})$ Lie algebra,

$$\begin{aligned}
[J_3,J_\pm]_\star &= \pm\,2i\,J_\pm\,,\\
[J_+,J_-]_\star &= i\,J_3\,.
\end{aligned} \tag{5.18}$$

The closure of the inner automorphisms at quadratic order can also be seen at the level of the corresponding group elements

$$\mathcal{Q}(\Delta,\xi)(x)=\det\left(\mathbb{1}_d-\Delta\,\theta\,\Delta\,\theta\right)^{1/4}\,\exp\left(i\,\Delta^{ij}\,x_i\,x_j+i\,\xi\cdot x\right), \tag{5.19}$$

where $\xi^i\in\mathbb{R}$ and Δ is an invertible, symmetric real-valued $d\times d$ matrix. The gaussian functions (5.19) are $\star$-unitary, and we have used the fact that the $\star$-exponential of a quadratic form on $\mathbb{R}^d$ can be written as an ordinary gaussian function [35]. To compute the generalization of the projective translation representation (4.30), we use the Fourier integral kernel representation of the $\star$-product (2.4) [58]

$$(f\star g)(x)=\frac{1}{\pi^d\,|\det(\theta)|}\iint \mathrm{d}^dy\;\mathrm{d}^dy'\;f(y)\,g(y')\;\,\mathrm{e}^{\,-2i\,(\theta^{-1})^{ij}\,(x-y)_i(x-y')_j}\,, \tag{5.20}$$

which follows directly from the Weyl representation (3.6), (3.7) and (3.8). By gaussian integration it yields the group composition law

$$\mathcal{Q}(\Delta,\xi)\star\mathcal{Q}(\Delta',\xi')=\Xi\left(\Delta,\Delta';\xi,\xi'\right)\mathcal{Q}\left(\Lambda(\Delta,\Delta')\,,\,\lambda(\Delta,\Delta';\xi,\xi')\right), \tag{5.21}$$

where

$$\Xi(\Delta,\Delta';\xi,\xi') = \exp\left[\frac{1}{4}\,\mathrm{tr}\ln\left(\frac{\mathbb{1}_d - \Delta'\,\theta\,\Delta'\,\theta}{\mathbb{1}_d + \Delta\,\theta\,\Delta'\,\theta}\right) + \frac{i}{4}\left(\frac{1}{\Delta}\right)_{ij}\,\xi^i\,\xi^j - \right.$$
$$\left. - \frac{i}{4}\left(\frac{1}{(\mathbb{1}_d + \Delta\,\theta\,\Delta'\,\theta)\,\Delta}\right)_{kl}\,(\mathbb{1}_d - \Delta\,\theta)_i{}^k\,(\mathbb{1}_d + \Delta\,\theta)_j{}^l\,\xi'^i\,\xi'^j\right],$$
$$\Lambda(\Delta,\Delta') = -\frac{1}{\theta\,\Delta\,\theta} + (\mathbb{1}_d - \Delta\,\theta)\,\frac{1}{(\mathbb{1}_d + \Delta'\,\theta\,\Delta\,\theta)\,\theta\,\Delta\,\theta}\,(\mathbb{1}_d + \Delta\,\theta)\,,$$
$$\lambda(\Delta,\Delta';\xi,\xi') = -\frac{1}{\Delta\,\theta}\,\xi - (\mathbb{1}_d - \Delta\,\theta)\,\frac{1}{(\mathbb{1}_d - \Delta'\,\theta\,\Delta\,\theta)\,\Delta\,\theta}\,(\mathbb{1}_d + \Delta\,\theta)\,\xi'\,. \tag{5.22}$$

However, at higher orders in the x_i's, whereby the Moyal and Poisson brackets no longer coincide, this property does not generally hold anymore. This again owes to the complicated nature of the deformed completion $\mathcal{C}_\theta^*$ which defines the gauge algebra (4.46). The geometrical, diffeomorphism Lie subalgebras arise only in the symplectic limit whereby the gauge symmetries are associated with isometries of the flat space $\mathbb{R}^d$. Indeed, the $\star$-product (2.4) is only generically invariant under canonical transformations of the spacetime, since these are precisely the diffeomorphisms which preserve the Poisson bi-vector $\partial_\xi \wedge \partial_\eta$ on $\mathbb{R}^d$ that defines (2.4). This is the *only* sense in which diffeomorphisms are realizable as inner automorphisms on a Moyal space.

5.2 Higher order algebras and UV/IR mixing

Let us now look at subalgebras generated by more complicated combinations of the basis elements of $\mathrm{u}(\mathcal{A}_\theta)$. For simplicity we shall again only consider the two-dimensional case. Now, it is possible to use a more involved structure and find *undeformed* versions of the Lie algebras encountered in the previous subsection. For example, it is readily checked that the operators

$$L_{12} = \frac{1}{\theta}\,T_{\vec{e}_2}\,, \qquad P_1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,T_{2n\vec{e}_1}\,, \qquad P_2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,T_{(2n+1)\vec{e}_1} \tag{5.23}$$

generate the standard Lie algebra of the euclidean group ISO(2), i.e. $[P_1,P_2]_\star = 0$. The construction can be straightforwardly generalized to d dimensions. This algebra has been described in [59] as a subalgebra of the Poisson-Lie algebra of phase space. In other words, for the functions (5.23), the Moyal and Poisson brackets again coincide, showing once more how the geometrical symmetries of non-commutative gauge theory are tied to the symplectic diffeomorphisms of spacetime.

To understand geometrically what the undeformed operators (5.23) represent, let us compute the corresponding inner automorphisms of the algebra $\mathcal{A}_\theta$. Using the translational property (4.29), we find their actions on a function $f \in \mathcal{A}_\theta$ to be given by

$$i\,[L_{12},f]_\star = \partial_1 f\,,$$

$$i\,[P_1,f]_\star = \left(\overrightarrow{\mathsf{d}}_1 f\right)\star\mathcal{T}(-\vec{e}_1) - \left(f\,\overleftarrow{\mathsf{d}}_1\right)\star\mathcal{T}(\vec{e}_1)\,,$$
$$i\,[P_2,f]_\star = -i\,\left(\overrightarrow{\mathsf{d}}_1 f\right)\star\mathcal{T}(-\vec{e}_1) - i\,\left(f\,\overleftarrow{\mathsf{d}}_1\right)\star\mathcal{T}(\vec{e}_1)\,, \tag{5.24}$$

where generally the forward and backward shift operators $\overrightarrow{\mathsf{d}}_i$ and $\overleftarrow{\mathsf{d}}_i$ are defined by

$$\overrightarrow{\mathsf{d}}_i\,f(x) = \frac{1}{2}\,\left(f(x+\theta\,\hat{\imath}) - f(x)\right),$$
$$f(x)\,\overleftarrow{\mathsf{d}}_i = \frac{1}{2}\,\left(f(x) - f(x-\theta\,\hat{\imath})\right), \tag{5.25}$$

with $\hat{\imath}$ a unit vector in the $i^{\rm th}$ direction of spacetime. From (5.24) we see that this undeformed representation of the euclidean algebra acts in an unusual geometric way. The "rotation" L_{12} acts by translations in the x_1 direction. In other words, the representation (5.23) selects a choice of axes in which it generates non-compact rotations. The two "translations" P_1 and P_2 affect a *lattice* displacement in the x_2 direction, with lattice spacing determined by the deformation parameter θ. The operators $\overrightarrow{\mathsf{d}}_i$ and $\overleftarrow{\mathsf{d}}_i$ are, up to a factor of the lattice spacing, the corresponding discrete derivatives. In fact, the transformations (5.25) are reminiscent of the shift operator representation of the corresponding Heisenberg-Weyl group elements. This property may again be attributed to the mixing of ultraviolet and infrared scales in non-commutative quantum field theory. Namely, the local, infinitesimal translations induce finite, discrete shifts by the scale of noncommutativity, and we recover a continuum interpretation of the UV/IR mixing that occurs in the non-perturbative lattice regularization of non-commutative field theories [7]. The unusual nature of these point symmetries is a consequence of the inherent non-locality of the gauge theory.

A similar property holds for the Poincaré algebra. The extra generators are given by

$$K_{12} = \frac{1}{\theta}\,T_{\vec{e}_2}\,, \qquad P_1 = \sum_{n=0}^\infty \frac{1}{n!}\,T_{n\vec{e}_1}\,,$$
$$P_2 = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\,T_{n\vec{e}_1}\,, \tag{5.26}$$

and they realize the undeformed Poincaré algebra iso$(1,1)$. Here the "boost" K_{12} acts by translations in the x_1 direction, so that the lines of constant x_1 may be interpreted as different inertial reference frames. The space and time translations P_1 and P_2 correspond to discrete shifts by the noncommutativity scale along the backward and forward light-cone directions, respectively.

Evidently there are many other examples that can be constructed along similar lines, in which a set of transformations has the algebraic structure of a known transformation group of the spacetime (or some other Lie group). The general feature will always be the same. Either the operators act canonically on the spacetime coordinates but yield deformations of the commutation relations of the Lie algebra, or else they realize the Lie algebra exactly but display highly non-local effects that make their geometrical interpretations differ enormously from the anticipated ones. We shall see in the next section how these unusual effects come up again in the gauge transformations of the fields. In all of these cases the exotic behaviours are the characteristic properties of the spacetime symmetries induced by non-commutative gauge theories. Indeed, the properties unveiled here illustrate that it is not entirely correct to regard a non-commutative gauge theory as an ordinary one on a commutative space in which a non-commutative algebra is defined.

5.3 Infinite-dimensional algebras

We will now describe some simple infinite-dimensional subalgebras of $\mathrm{u}(\mathcal{A}_\theta)$. First of all, there are d obvious abelian subalgebras, generated by $T_{n\vec{e}_i}$, $n \in \mathbb{Z}_+$, for each $i = 1, \ldots, d$. These algebras yield gauge transformations by functions which depend only on a single coordinate x_i and correspond to a choice of "configuration" subspace of non-commutative $\mathbb{R}^d$. However, such particular choices are merely a matter of a choice of axes in $\mathbb{R}^d$. More generally, changing basis we find that the linear combinations of the form

$$C_n(\vec{c}\,) = \sum_{i=1}^{d} c_i \, T_{n\vec{e}_i} \, , \qquad n \in \mathbb{Z}_+ \, , \tag{5.27}$$

generate an infinite-dimensional abelian subalgebra of $\mathrm{u}(\mathcal{A}_\theta)$ for all *constant* real-valued d-vectors $\vec{c}$. Thus there actually exists a continuous d-parameter family of such sorts of abelian subalgebras.

Formally, abelian subalgebras are parametrized by a lagrangian submanifold of the underlying symplectic space, i.e. a subspace on which the symplectic form vanishes, and correspond to a foliation into symplectic leaves on each of which the Casimir one-form of the Lie algebra $\mathrm{u}(\mathcal{A}_\theta)$ vanishes. A more non-trivial choice of lagrangian submanifold corresponds to the $2^{d/2}$ classes of "diagonal" subalgebras of $\mathrm{u}(\mathcal{A}_\theta)$. In each commuting two-dimensional skew block a, these are generated by the operators $T_{\underline{n}}^{(a)}$, $n \in \mathbb{Z}_+$, which according to (3.17) have the property

$$\left[T_{\underline{n}}^{(a)} \, , \, T_{\underline{m}}^{(a)}\right]_\star = 0 \, . \tag{5.28}$$

Closely related to this subalgebra is the one generated by the odd Wigner functions $\Sigma_{\underline{n}}^{(a)-}$, $n \in \mathbb{Z}_+$, in the density matrix basis. It consists of radially symmetric functions, i.e. those which depend only on the real variables $|z_a|$, and it is generated by the completion of the Cartan subalgebra of the infinite unitary Lie algebra $\mathrm{u}(\infty)$. Although similar, these two abelian algebras are not the same, for example it is easy to check by using (3.37) that their generators do not mutually commute.

These abelian subalgebras can each be regarded as the local symmetry algebra of a *commutative* $\frac{d}{2}$-dimensional U(1) gauge theory. This opens up the possibility of a mechanism of dimensional reduction for which the gauge theory on a d-dimensional non-commutative space has a vacuum which is invariant only under the gauge transformations corresponding to an ordinary gauge theory on a $\frac{d}{2}$-dimensional commutative subspace. The target space will, however, manifest its true nature at energies of order $\theta^{-1/2}$ whereby, in the string picture, the effects of the background B-field become important. Considering the complete reduction of the non-commutative dimensions may lead to a sort of universal gauge theory, containing all Yang-Mills theories, along the lines described in [60]. We shall describe some consequences of this simple observation in the next section.

Finally, let us point out that there is a set of d non-abelian infinite-dimensional subalgebras generated by the $T_{\vec{n}}$'s with a particular component integer n^i of odd parity. This follows trivially from the fact that the sum of three odd integers (n^i, m^i and $2p+1$ in (3.17) is odd. The even/odd parity of n^i for each $i = 1, \ldots, d$ gives an extra $\mathbb{Z}_2^d$ grading to the $\star$-monomial representation of $\mathrm{u}(\mathcal{A}_\theta)$ which is related to the $\pm$ grading of the density matrix basis as described in section 3.3. It would be interesting to construct a genuine $W_{1+\infty}$ subalgebra of $\mathrm{u}(\mathcal{A}_\theta)$ in this setting, which at present only appears as the approximate symmetry algebra (4.10) at order θ_{ij}.

6. Applications

In this section we will briefly describe some implications of the analysis of the previous sections by applying the formalism to study some of the physical characteristics of non-commutative gauge theories. First we will discuss some further aspects of the relationships between inner automorphisms, gauge transformations and spacetime diffeomorphisms. Then we will describe some aspects of the geometrical structure of solitons in non-commutative gauge theory.

6.1 Geometry of non-commutative gauge transformations

For the bulk of this paper thus far, we have focused on the geometrical interpretation of inner automorphisms of the algebra $\mathcal{A}_\theta$, without saying what the implications are for the corresponding inhomogeneous gauge transformations in (2.13). We are now ready to lend some geometrical insights into the nature of the local $\star$-gauge symmetry of non-commutative Yang-Mills theory. For this, we expand a generic non-commutative gauge field $A_i(x)$ over $\mathbb{C}[[x_1, \ldots, x_d]]/\mathcal{R}_\theta$ in terms of the momentum operators (5.3) as [4]

$$A_i \equiv -\sum_{j=1}^{d} \theta_i^{\ j}\, T_{\vec{e}_j} + \Pi_i \,. \tag{6.1}$$

An essential feature of gauge theory on a non-commutative space is the gauge invariance of the spacetime coordinates up to a transformation of the form

$$\delta_\lambda x_i = c_i + \Lambda_i^{\ j}\, x_j \,, \tag{6.2}$$

where c_i are constants parametrizing the (trivial) center $\mathbb{C}$ of the algebra, and $\Lambda \in \mathrm{Sp}(d)$ are constant matrices which parametrize the group of rotations that leaves the noncommutativity parameters invariant, i.e.

$$\Lambda_i{}^k \, \theta_{kl} \, \Lambda_j{}^l = \theta_{ij} \, . \tag{6.3}$$

Only in that case is the algebra (2.1) and 2.2) preserved by the gauge transformation. Unless specified otherwise, we shall take $c_i = \Lambda_i{}^j = 0$ in (6.2) for simplicity.

The fields Π_i which are defined by (6.1) may be thought of as gauge covariant momentum operators, because the gauge transformation rule (2.13) is equivalent to the infinitesimal inner automorphism

$$\delta_\lambda \Pi_i = i \, [\lambda, \Pi_i]_\star \, . \tag{6.4}$$

Moreover, the entire non-commutative gauge theory can be expressed in terms of the new fields Π_i. The gauge covariant derivative (2.30) may be written as the inner derivation

$$\nabla_i^{\rm ad} f = i \, [f, \Pi_i]_\star \, , \qquad \forall f \in \mathcal{A}_\theta \, , \tag{6.5}$$

while the field strength tensor (2.11) is the sum

$$F_{ij} = -i \, [\Pi_i, \Pi_j]_\star + B_{ij} \, . \tag{6.6}$$

The classical vacua of the non-commutative Yang-Mills theory (2.10), i.e. the flat non-commutative gauge connections, are in this setting the non-commuting covariant momenta with $[\Pi_i, \Pi_j]_\star = -i \, B_{ij}$. In the D-brane picture, the corresponding global minima are identified with the closed string vacuum possessing no open string excitations. The transition from the momentum operators P_i to the covariant ones Π_i reflects the bi-module structure based on the non-commutative function algebras with deformation parameters θ and $-\theta$, and it is the basis of the relation between the commutative and non-commutative descriptions of the same theory [54]. This gives a dynamical interpretation to the noncommutativity of spacetime.

Thus the local non-commutative gauge symmetry is determined entirely by the inner automorphisms (6.4), and in this way the geometrical interpretations given in the previous section carry through to the non-commutative gauge fields. Let us now examine these transformations in some detail. The simplest ones are the spacetime translations generated by the operators (5.3). Notice first that although the functions (5.3) only form a projective representation of the d-dimensional translation group, the corresponding Lie algebra (2.14) of gauge transformations forms a true representation,

$$\left[\delta_{P_i} \, , \, \delta_{P_j}\right] = 0 \, , \tag{6.7}$$

because the projective phase in (5.5) lies in the center $\mathbb{C}$ of the algebra $\mathcal{A}_\theta$. The emergence of this true representation of the translation group owes to the effective enlargement of the deformed group representation via the decomposition (6.1).

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Namely, the gauge transform (2.13) in this case reads

$$\delta_{P_j} A_i = \partial_j A_i + B_{ij} \ , \tag{6.8}$$

which up to the addition of the magnetic flux B_{ij} is the anticipated transformation rule for a one-form field under spacetime translations. Again (6.8) corresponds to a projective representation of the translation group. This is yet another manifestation of UV/IR mixing, in that a translation along a direction j causes a constant shift of the gauge fields A_i along the orthogonal non-commutative directions i. Constant shifts $A_i \mapsto A_i + c_i$ of the gauge fields correspond to global gauge transformations by the center of the algebra, and are therefore global symmetries of the field theory (2.10). Indeed, the field strength tensors (2.11) are invariant under such shifts. The fact that a local gauge transformation induces a global symmetry is again due to the non-locality of the non-commutative gauge theory.

Let us now consider the rotation generators (5.7). We immediately encounter two important differences from the case above. First, the inhomogeneous terms in (2.13) that appear in the non-commutative directions are the functions $\theta^{-1}\, T_{\vec{e}_i}$, which are non-central elements of the algebra. This owes to the more complicated group composition law (5.21) and 5.22) for gaussian functions, or equivalently to the non-central extensions of the deformed iso(d) algebras (5.12). Second, the corresponding inner automorphism (6.4) generates a rotation of a *scalar* field, not a one-form field. In other words, the non-commutative gauge symmetry ignores the vector index of the non-commutative gauge fields. The reason for this will be explained below. Analogous but more involved properties are true of the generators (5.23). In that case, the corresponding gauge transformations (5.24) (along with the deformed translational symmetry) immediately imply the inherent non-locality of non-commutative Yang-Mills theory, i.e. that gauge invariant observables are necessarily non-local. These structures all come about from the mixing of colour and spacetime symmetries that we have alluded to earlier.

By using the covariant momentum operators we may also examine to what extent a generic local diffeomorphism of $\mathbb{R}^d$ can be realized as a genuine gauge symmetry of the non-commutative Yang-Mills theory. Given an arbitrary, local vector field $V = V^i(x)\,\partial_i$ on $\mathbb{R}^d$, we introduce a corresponding gauge function $\lambda = \lambda_V$ (again over $\mathbb{C}[[x_1, \ldots, x_d]]/\mathcal{R}_\theta$) by the $\star$-anticommutator

$$\lambda_V = -\frac{1}{2} \sum_{j=1}^{d} \theta_i^{\ j} \ \left\{T_{\vec{e}_j} \, , \, V^i\right\}_\star . \tag{6.9}$$

Then the corresponding gauge transformation (6.4) can be expanded in the deformation parameters by using (2.9) and (3.32) to obtain the leading order result

$$\delta_{\lambda_V} \Pi_i = V^j\, \partial_j \Pi_i + \sum_{j=1}^{d} \theta_k^{\ j}\, T_{\vec{e}_j}\, \delta^{lm}\, \delta^{np}\, \theta_{mp}\, \partial_l V^k\, \partial_n \Pi_i + O(\theta_{ij}) \, . \tag{6.10}$$

The first term in (6.10) is close to the expected transformation law for Π_i under an infinitesimal diffeomorphism, except that it treats it as a scalar field. This is not surprising, because the gauge theory we have formulated is defined on a flat space and so only possesses a *global* Lorentz symmetry, and not a local one. In other words, only those diffeomorphisms which are isometries of flat spacetime arise in non-commutative gauge transformations. This feature is described further in [15], and possible extensions of non-commutative Yang-Mills theory to incorporate local frame independence are analysed in [15, 17]. In any case, this term is accompanied by the second term in (6.10) which is of the same order in θ_{ij}. This fact on its own prevents one from realizing arbitrary diffeomorphisms in terms of $\star$-gauge symmetries. One could nonetheless carry on and attempt to interpret the first term in (6.10) as a vierbein transformation rule for a flat space vierbein field h_i^j, as in [17], defined through the decomposition $\Pi_i = \sum_j \theta_k{}^j \, T_{\vec{e}_j} \, h_i^k$. But this is only possible when the spacetime coordinates themselves gauge transform as $\delta_{\lambda_V} x_i = V_i(x)$. Unless the vector field V is parametrized by an element of the Lie algebra $\mathbb{C} \oplus sp(d)$ as in (6.2), such a transformation will map non-commutative $\mathbb{R}^d$ onto a different non-commutative space and will not be a symmetry of the theory. The resultant space would be related to the dynamics of D-branes in non-constant B-fields, which are described in terms of non-associative algebras [61]. Such a manipulation will thereby only work for the gaussian spacetime transformations that we described in section 5.1. Again this is just an indication of the basic fact that only (deformations of) symplectomorphisms are realized as non-commutative gauge transformations.

6.2 Conformal non-invariance

In previous sections we have seen how gauge transformations are given as inner automorphisms of the algebra $\mathcal{A}_\theta$ and that they correspond to symplectic diffeomorphisms. We can also use the present formalism to investigate this geometrical feature in a bit more detail. Not all inner automorphisms of $\mathcal{A}_\theta$ correspond to gauge transformations, but only the ones which are generated by hermitean, Schwartzian elements. It is also straightforward to show explicitly that there exist diffeomorphisms of $\mathbb{R}^d$ which cannot be realized via $\star$-commutators. For example, let us consider the scale transformation $x_i \mapsto x_i + \alpha_i \, x_i$, $i = 1, \ldots, d$ (no sum on i), with α_i real-valued constants. To realize this coordinate transformation as an infinitesimal inner automorphism of $\mathcal{A}_\theta$, we seek an element S_α of the algebra with the properties

$$S_\alpha = \sum_{\vec{m} \in \mathbb{Z}_+^d} s_{\vec{m}} \, T_{\vec{m}} \,,$$

$$i \, [S_\alpha, T_{\vec{n}}]_\star = \sum_{i=1}^d \alpha_i \, n^i \, T_{\vec{n}} \,, \qquad \forall \vec{n} \in \mathbb{Z}_+^d \,. \tag{6.11}$$

Taking $\vec{n} = \vec{e}_1$ in (6.11), and using (3.18), gives $s_{\underline{1}} = \alpha_1$ and $s_{\vec{m}} = 0 \quad \forall \vec{m} \neq \underline{1}$. On the other hand, taking $\vec{n} = \vec{e}_2$ in (6.11) gives $s_{\underline{1}} = -\alpha_2$, which generically leads to a contradiction. This proves that conformal transformations cannot be obtained via inner automorphisms of $\mathcal{A}_\theta$ and do not constitute gauge symmetries of non-commutative Yang-Mills theory. Notice that this argument is independent of any reality assumption or asymptotic behaviour of the expansion coefficients $s_{\vec{m}}$ of (6.11).

The lack of a conformal gauge symmetry can also be seen algebraically as follows. In the skew-diagonalization, the truncated Lie algebra defined by the first line of (4.8) also contains, in each skew-block, an infinite-dimensional subalgebra which corresponds to the positive Borel subalgebra of the classical Virasoro-Witt algebra. By defining

$$\begin{aligned} \ell_n &= \frac{1}{\theta} T_{n\vec{e}_1+\underline{1}}\,, \qquad n \geq 0\,, \\ \ell_{-1} &= \frac{1}{\theta} T_{\vec{e}_2} \end{aligned} \tag{6.12}$$

in the two-dimensional case, we arrive at the $\star$-commutation relations

$$[\ell_n, \ell_m]_\star = i\,(n-m)\,\ell_{n+m}\,. \tag{6.13}$$

This algebra contains an su(2) Lie subalgebra generated by the functions ℓ_0 and $\ell_{\pm 1}$. It is possible to generate a full Virasoro-Witt algebra by defining instead the operators

$$L_n = \begin{cases} \frac{1}{\theta}\left(T_{n\vec{e}_1+\underline{1}} + \frac{i\,\theta}{2\;n!}\,T_{n\vec{e}_1}\right), \qquad n \geq 0\,, \\ \frac{1}{\theta}\,T_{\vec{e}_2}\,, \qquad n=-1\,, \\ \frac{1}{\theta}\sum_{k=0}^\infty\;\sum_{l=0}^k \begin{pmatrix} k \\ l \end{pmatrix} \Big(T_{(|n|-1)(k-l)\vec{e}_1+\underline{1}} \\ +\frac{i\,\theta}{2\;((|n|-1)(k-l))!}\,T_{((|n|-1)(k-l)-1)\vec{e}_1}\Big)\,, \qquad n<-1\,. \end{cases} \tag{6.14}$$

The third expression in (6.14) is to be understood as a formal series, because it strictly speaking only makes sense, as a Weyl operator say, on the domain of Hilbert space in which the eigenvalues of the hermitean operator $\hat{x}_1$ lie in the subset $\mathbb{R}_- \cup [2^{1/n}, \infty)$ of the real line. Then, by using (3.17) we may infer the $\star$-commutation relations

$$[L_n, L_m]_\star = i\,(n-m)\,L_{n+m}\,. \tag{6.15}$$

Interchanging the unit lattice vectors $\vec{e}_1$ and $\vec{e}_2$ in (6.14) defines corresponding "anti-holomorphic" generators $\overline{L}_n$. It may be straightforwardly checked that the operators (6.14) (along with the $\overline{L}_n$) generate the anticipated conformal transformations of scalar fields in two dimensions via infinitesimal inner automorphisms. However, although we have obtained the correct commutation relations of the conformal algebra in two dimensions, the realization (6.14) defines a non-unitary representation of this Lie algebra in the Moyal bracket. In addition, $[L_n, \overline{L}_m]_\star \neq 0$, so that the two copies of the algebra are not independent. This demonstrates again that there is no unitary realization of conformal transformations as non-commutative gauge symmetries on the Moyal space.

6.3 Geometrical structure of non-commutative solitons

A concrete realization of the breaking of the $\mathrm{u}(\mathcal{A}_\theta)$ gauge symmetry to the infinite-dimensional commutative subalgebras described in section 5.3 is provided by the soliton configurations of non-commutative field theory [62]. They are determined by projection operators or partial isometries in the unital algebra $\mathcal{A}_\theta^+ = \mathcal{A}_\theta \oplus \mathbb{C}$. For simplicity, we shall again restrict to the two-dimensional case. For $\mathrm{U}(N)$ non-commutative gauge theory, the static soliton solutions are given by the covariant momentum operators [22, 63]

$$\Pi_i = \mathsf{S}\, T_{\vec{e}_i}\, \mathsf{S}^\dagger + \sum_{\mu=1}^{m} \alpha_i^\mu\, \Sigma_{\underline{\mu}}^- \tag{6.16}$$

for $i = 1, 2$, where the density matrices $\Sigma_{\underline{\mu}}^-$, $\mu = 1, \ldots, m$, correspond to an m-dimensional subspace of the Hilbert space $\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^N$, and S is the associated shift operator which is the partial isometry in $\mathbb{M}_N(\mathcal{A}_\theta^+)$ obeying

$$\mathsf{S}^\dagger \star \mathsf{S} = \mathbb{1}\,, \qquad \mathsf{S} \star \mathsf{S}^\dagger = \mathbb{1} - \sum_{\mu=1}^{m} \Sigma_{\underline{\mu}}^-\,. \tag{6.17}$$

The $2m$ moduli parameters α_i^μ of the classical gauge field configuration (6.16) may be interpreted as the locations of D0-branes (the solitons) inside D2-branes (the non-commutative transverse space), with m and N the 0-brane and 2-brane charges, respectively.

To understand better the soliton solutions (6.16) within the present context, we will study the somewhat simpler case of static solitons in a non-commutative *scalar* field theory in the limit of large noncommutativity θ. As is well known [62], the space of such solutions is spanned, for a given scalar potential, by an orthonormal basis of projectors P_n, $n \in \mathbb{Z}_+$, in the algebra $\mathcal{A}_\theta^+$,

$$\begin{aligned} \mathsf{P}_n \star \mathsf{P}_m &= \delta_{nm}\, \mathsf{P}_n\,, \\ \sum_{n=0}^{\infty} \mathsf{P}_n &= 1\,. \end{aligned} \tag{6.18}$$

Using the Weyl-Wigner correspondence (3.8) and the density matrix basis (3.25), it is easy to see that a basis of functions satisfying (6.18) is given by

$$\mathsf{P}_n = \frac{1}{2}\, \Sigma_{\underline{n}}^-\,. \tag{6.19}$$

By using (2.26), we see that the symmetry group of eqs. (6.18) is precisely the unitary group $\mathrm{U}(\mathcal{A}_\theta^+)$. Each non-vanishing solution $\mathsf{P}_n(|z|)$ breaks this symmetry down to a commutative subgroup. A rank m projection operator will induce a broken $\mathrm{U}(m)$ subgroup of $\mathrm{U}(\mathcal{A}_\theta^+)$ corresponding to the symmetry group of m coincident D-branes.

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Inner automorphisms of the algebra $\mathcal{A}_\theta^+$ rotate the basis (6.19) to generically non-radially symmetric soliton solutions. For $\theta < \infty$, the kinetic term for the scalar field will explicitly break the unitary gauge symmetry down to the euclidean subgroup ISO(2) of $\mathrm{U}(\mathcal{A}_\theta^+)$. This lifts the manifold of soliton solutions to a discrete set of solutions.

The non-trivial structure of the gauge group can be seen explicitly in this context by approximating these soliton solutions by the finite-rank operators constructed in section 4. Given the Cartan and Weyl bases (4.21) and (4.35), we may use a discrete version of the Wigner transform (3.7) to define the functions

$$E_{\vec{n}}(x) = \frac{1}{N} \sum_{\vec{m} \in \mathbb{Z}_N^2} \mathrm{e}^{\,2\pi i\,\vec{m}\cdot x/N} \ \mathrm{Tr}_N\left(\hat{E}_{\vec{n}}\,\hat{\mathcal{W}}_{\vec{m}}^{(N)}\right), \tag{6.20}$$

where $\vec{n} \in \mathbb{Z}_N^2$ and here x_i are interpreted as coordinates on a periodic, square lattice of spacing $\frac{1}{N}$. This corresponds to a finite-dimensional representation of the Heisenberg-Weyl group. By using the change of basis (4.43), the orthonormality relations (4.40), and the lattice completeness relations

$$\frac{1}{N} \sum_{\vec{m} \in \mathbb{Z}_N^2} \mathrm{e}^{\,2\pi i\,\vec{m}\cdot(x-y)/N} = N^2\,\delta(x-y)\,, \tag{6.21}$$

the Wigner functions (6.20) can be written as

$$E_{\vec{n}}(x) = -4\pi M N \omega^{(n^1-n^2)(n^1-2n^2-1)/2}\,\mathrm{e}^{\,-2\pi i(n^1-n^2)x_2/N}\,\delta\left(x_1 + M(n^1-2n^2)\right). \tag{6.22}$$

We may thereby approximate the solutions of the non-commutative soliton equations (6.18) by the semi-localized finite-rank projection operators

$$\mathsf{P}_n^{(N)}(x) = E_{\underline{n}}(x) = -4\pi M N\,\delta(x_1 - Mn)\,. \tag{6.23}$$

Such fuzzy soliton configurations have also been obtained in [64]. Note that in this lattice basis the coordinate x_1 plays a role analogous to the continuum radial coordinate $|z|$. This is expected from the properties of the rotation generators described in section 5.2, namely that the true representation of the euclidean group will affect rotations in the x_1 coordinate, and hence of the soliton.

The discrete projectors (6.23) illustrate the non-trivial moduli space of soliton solutions that arise, which are parametrized by their location x_1. They describe stripes on the Moyal plane labelled by the integers M and n. However, they do not converge to the continuum projector solutions in the large-N limit, which are given in terms of Laguerre polynomials as in section 3. This illustrates the basic importance of the large-N completion in (4.46) that defines the unitary group of the algebra $\mathcal{A}_\theta^+$, and it can be understood in terms of its corresponding K-theory [26]. While the group $\mathrm{K}_0(\mathcal{A}_\theta^+)$ is given in terms of partial-unitary equivalence classes of projectors in $\mathcal{A}_\theta^+ \otimes \mathbb{M}_\infty(\mathbb{C})$, the K_1-group is given in terms of the connected components of the gauge group as

$$\mathrm{K}_1\left(\mathcal{A}_\theta^+\right) = \frac{U\left(\mathcal{A}_\theta^+\right)}{U\left(\mathcal{A}_\theta^+\right)_0}\,, \tag{6.24}$$

where $\mathrm{U}(\mathcal{A}_\theta^+)_0$ denotes the (local) connected subgroup of $\mathrm{U}(\mathcal{A}_\theta^+)$. With this definition it is evident that the K_1-group of any finite-dimensional matrix algebra $\mathbb{M}_N(\mathbb{C})$ is trivial, and, since K-theory is stable under inductive limits, so is that of $\mathbb{M}_\infty(\mathbb{C})$ as defined in section 4.2. On the other hand, the group (6.24) is known to be non-trivial in certain instances. Therefore, an appropriate completion of the infinite unitary group $\mathrm{U}(\infty)$ is required to preserve the K-theoretic properties of the Moyal space. Such completions in the case of the non-commutative torus are described in [49].

Of course, the applications described in this subsection require an analysis of the *global* gauge group, as in (6.24), which lies beyond the scope of this paper. They are important however for many of the applications of these results to the description of D-branes as solitons in the effective field theory of open strings. Indeed, D-brane charges are classified by the K-theory groups of the spaces on which they are defined [65, 66]. They correspond to projectors or partial isometries arising as solitonic lumps of the tachyon field in a given non-BPS system of higher-dimensional D-branes [62]. The mechanisms for symmetry breaking described in section 5.3 can thereby lend a more geometrical picture to processes involving tachyon condensation, and, in particular, the low-energy effective string field theory action [67] may in this setting correspond to some sort of universal gauge theory [60]. In this context, the automorphism group (4.14) corresponds to the non-commutative gauge group of D-branes in the presence of NS5-branes [68], i.e. in a topologically non-trivial B-field, which may be realized as certain twisted $\mathrm{PU}(\mathcal{H})$ bundles [66, 69]. The corresponding description of the D-brane charges is known to be given by a twisted version of K-theory [66, 69].

7. Conclusions

The intriguing mixing between the infrared and ultraviolet limits is a characteristic feature of gauge theories on non-commutative spaces, and it is intimately tied to the mixing between spacetime variables and gauge degrees of freedom. This is reflected in the gauge transformations of the fields which drastically alter not only their internal degrees of freedom, but also their spacetime dependence. It is a consequence of the fact that, strictly speaking, it does not makes sense to speak of "point dependence" for these field theories, as the concept of a point is ill-defined. In particular, it is not possible to regard the gauge algebra as the tensor product of a finite-dimensional algebra by the set of functions on a point. This implies that, like in general relativity, the spacetime structure is a gauge non-invariant concept. We have seen that the gauge degrees of freedom come from a deformation of the Poisson-Lie algebra of symplectic diffeomorphisms. These gauge transformations are intimately related to unitary conjugation by elements of the group $\mathrm{U}(\infty)$, which is the natural symmetry group that arises from the large-N matrix models which provide non-perturbative regularizations of non-commutative gauge theory.

It is tempting at this point to speculate on the relationships between these deformed canonical transformations and time evolution. A dynamical system can be thought of as a C^*-algebra together with a one-parameter group of symplectic automorphisms which is generated by a hamiltonian. In the non-commutative case, gauge transformations are equivalent to canonical transformations of spacetime. It would be interesting to interpret this deformed $\mathrm{U}(\infty)$ gauge symmetry in terms of the group of hamiltonian flows.

The sort of non-commutative geometry discussed in this paper is of course but an approximation to the full string theory. The general structure of spacetime at the string (or Planck) scale is likely to be described by the $*$-algebra $\mathcal{A}_{\rm str}$ of open string fields which is defined by the gluing together of strings. In the low-energy scaling limit under consideration, this algebra factorizes as [67] $\mathcal{A}_{\rm str} \to \mathcal{A}^0_{\rm v} \otimes \mathcal{A}_\theta$, where $\mathcal{A}^0_{\rm v}$ is the algebra of vertex operators with vanishing momentum along the non-commutative directions, and $\mathcal{A}_\theta$ is the non-commutative function algebra considered in this paper. If the deformed space we have dealt with is an accurate description of spacetime at energies of the order $\theta^{-1/2}$, then at these energies gauge invariance transforms (via deformed symplectic diffeomorphisms) the very structure of spacetime. Given that, in the non-commutative geometry based on closed string vertex operator algebras [10], generic diffeomorphisms of spacetime can be viewed as gauge transformations, it is possible that the stringy part of the algebra $\mathcal{A}_{\rm str}$ reinstates full general covariance as a genuine gauge symmetry. In the low-energy limit above, a spacetime diffeomorphism then has a "gauge part" related to the spacetime algebra $\mathcal{A}_\theta$ and a "conformal part" related to the stringy algebra $\mathcal{A}^0_{\rm v}$.

It is also possible that the effective string field theory action will only be invariant under a subset of the possible gauge transformations, leading to theories for which the accessible gauge theory is much smaller. The resultant gauge symmetry may either still be infinite dimensional, corresponding to some sort of dimensional reduction, or it may become finite dimensional, corresponding to a total dimensional reduction (along the non-commutative directions) with induced internal degrees of freedom. In this respect we could see the emergence of $\mathrm{u}(N) \subset \mathrm{u}(\infty)$ gauge models, which are otherwise difficult to obtain in gauge theories based on non-commutative geometry.

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